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NUMBER BY DEVELOPMENT

A METHOD
OF NUMBER INSTRUCTION

VOL. II.

Intermediate
Grades

BY

JOHN C. GRAY, A.M.
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PHILADELPHIA & LONDON
J. B. LIPPINCOTT COMPANY
1915

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JOHN C. GRAY, A.M.,
SUPERINTENDENT OF SCHOOLS, CHICOPEE, MASSACHUSETTS

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INTRODUCTION

AIMS

THE aim of this book is to furnish to the teacher and supervisor a working outline in detail of class work in common fractions in accordance with development ideals and suggestions in denominate numbers.

Development is the gradual unfolding of the subject of instruction by a process of natural, logical growth. It is an evolution, a building up process. It has to do with a teaching plan under which the subject to be taught is analyzed into its teaching units and these are so arranged that each in its place has definite relations with the product of previous steps and is necessary at that point in the sequence to the continuation of the building up process in that subject. It is a plan of work under which the aims to be realized are orderly mental habits, reasoning power, and clear imagery.

To know what the steps are in the teaching of a subject and to order these steps so that the teaching product shall be a body of organized knowledge instead of a series of disconnected items challenges the highest skill and demands long and open-minded experience in direct working contact with children of the age of those to whom the subject is to be taught.

The purpose of this book is to give the teacher outlines in detail of the subjects treated, the sequence of the teaching steps, the form of the objective work that is necessary for the establishment of clear and full imagery, and suggestions as to the method by which she may approach each successive step in each subject so as to afford the pupil the fullest opportunity to discover truth by inferences from previous experiences.

The details of sequence of teaching steps are not the only difficulties with which the teacher has to deal. There are details of suggestion for each step, methods of drawing pupils' attention to the line of development thought and of holding it there for a properly sustained consideration of the unit of truth involved. A teacher may have a perfectly organized plan of instruction in a given subject and yet be so unfortunate in her plan for leading up to the successive units that she *gives* to the pupils the facts or conclusions that they should and would discover for themselves by inferences. It is on this account that with the details of sequence those of suggestion have been included in order that the teacher working under plans new to her and slightly at variance with those with which she has been accustomed, may not be at loss to know how to proceed in an emergency.

THE GENERAL PLAN

The general plan of the work follows closely that used for introductory number in Volume I. This would necessarily be the case because number expressions and number operations do not change in meaning when passing from a quantity in which the measuring unit is a whole to a quantity in which it is one of the equal parts of a whole. The character of 6, for illustration, is the same whether we apply it as 6 or $\frac{6}{7}$ or .6. It is a group of six units considered together as *one*.

If the plan for introductory work in number develops imagery that is true, that same plan must of necessity be used for all forms of number or number operations, otherwise the imagery would become confused or be lost.

THE WORK OBJECTIVE

In the development of the plans of work here outlined the aim from the beginning has been to put the operations in each subject, so far as the subject would admit of objective representation, into objective form—to give the child the opportunity to construct them, to see the changes in quantity that they express.

Both of these subjects are fully objective. Decimal fractions, however, in Volume III will have no special objective work because they do not differ from common fractions in ways that constitute objective variations.

The author desires to thank the many loyal, intelligent, progressive teachers who have contributed to these outlines by experiment and suggestion. But for their assistance it would have taken many more years to complete them in the form in which they are here represented. His thanks are particularly due to Miss Mamie T. Leary, Principal of our Sheridan School, for many very valuable suggestions in the work in common fractions—critical experiments in the objective work, practical contributions to the oral work, and strong work in parallels in denominate numbers and general applied problems.

NUMBER BY DEVELOPMENT
A METHOD OF NUMBER INSTRUCTION

PART I

PRINCIPLES APPLICABLE IN NUMBER TEACHING

A. Principles in Development Theory

1. System in Teaching

That there is a right order and a wrong order of teaching steps in every subject is a matter that should require no discussion. It is true in every line of constructive work. If the machinist or the blacksmith desires to make a piece of steel into a fixture or tool or other article for a special use, economy in material and time demands that the various operations in the process of bringing that piece of steel to the desired form be so ordered that work done upon the material in one operation need not be undone or interfered with in the operations which follow. His ability to put system into his constructions is one of the tests of a mechanic's skill. In a larger way, the organization of a shop with reference to sequences of all kinds in the manufacturing work—sequences in machines, in operations of finishing, etc.—is one of its important problems of economical and efficient management.

Order and system are no less necessary in educational work. The very term development, unfolding, carries with it the presumption that there is a law of progress and growth; a logical first step and second step and succeeding steps; a plan that is consistently maintained; a single, unbroken line of progress from the beginning to the realization of the development aim.

The demand in the industries is for production—a higher and higher production for a given expenditure of labor. There is a similar demand upon the schools—a reasonable and healthful demand for a higher educational product without an increase in child-hour cost. System and order are obviously, therefore, conditions no less fundamental in the organization of school work under present day ideals in education than in the organization of the shop.

2. *The Training Value of Detailed Outlines*

a. THE TWO VIEWS AS TO OUTLINING

The question is sometimes asked in this matter of outlines of methods of teaching special subjects—written language, spelling, writing, but particularly number or objective number—whether an outline should go into details of presentation or give to the teacher only the general principles upon which the work should be based and leave to her the working out of these details—the units of instruction in each subdivision of the subject, the sequence of these elements, and the method in each case of presenting them. The whole matter turns on the question, Do the general interests of education demand that the teacher be unhampered as to the details of methods?

Some hold that a detailed system in any subject stifles; puts the teacher in a strait-jacket, as it were; tends to make the teacher's work formal and mechanical; and hinders progress in education by limiting the opportunities for originality in methods and for the contributions that freedom might stimulate.

Others take the view that practical schoolroom experience in the execution of detailed methods that are properly organized—organized on definite, logical plans—is a part of the necessary regular training of every teacher and that such training is essential to the exercise later of intelligent originality in methods of instruction. They believe that the teacher who has not had large experience in detailed work properly arranged cannot appreciate the need and the effect of separating a subject into its elements and arranging them in logical order preliminary to teaching it, and has not acquired, therefore, the point of view in teaching to make her an independent worker much less a contributor to her profession. Last of all is the conviction that the development of skill and power are in no way necessarily dependent upon the opportunities for and the exercise of originality.

b. Detailed Outlines Essential

There is a principle that is regarded as fundamental and is very strongly emphasized in training for school work, that the teacher who enters upon the teaching of a subject or subdivision of a subject without preparing in advance a most carefully detailed outline of the teaching steps and the method of approach for each step will waste in a greater or less degree her own and the pupils' time, to say nothing of the habits of inattention and loose thinking that her work will develop. It urges that every subject, every presentation, must have its instruction details prepared beforehand and entered in the general and the daily plan-book as the teacher's guide in the class work that is to follow.

c. ORDINARY OUTLINES PRODUCTS OF EXPERIENCE, NOT CREATIONS

When a teacher prepares for the work of a term and later the subdivisions of her subjects for daily work, she *organizes* rather than *creates*. If she is without schoolroom experience, a beginner, an outline in any given subject would manifestly be constructed from what she remembers of her experience as a pupil and from what she may be able to apply of her training experiences. She *creates* nothing.

If she has had schoolroom experience, subject plans and those for daily work are practically from the same source. They represent training experiences and those acquired through professional literature, conferences, and general contacts with other teachers. They embody new creations only as now and then a minor change in arrangement or method of approach is incorporated (or reincorporated, if the change is a revision of a conception that has advanced one or more stages) as an experiment which she has had in contemplation or has been working upon from term to term shaping details so that she may succeed in making sound theory square with child aptitudes and tendencies.

Plans, therefore, whether general or daily are in this way practically organizations of existing material. They may be enriched and broadened by wise and thoughtful experiments here and there, as has been suggested—experiments made in the hope of a higher quality of product—but changes of this kind, although in the nature of simple modifications, may require months and even years to mature.

d. PROFESSIONAL GROWTH FROM WORK NOT ORIGINAL

It has now been shown that the teacher's working plans are mainly organizations of old material—experiences acquired from those under whom she trained or that have come to her from other sources than original experimentation—and that originality enters very gradually into a teacher's work, developing by now one now another well conceived experiment that may have matured only after repeated revisions or modifications, each, perhaps, requiring a fresh group of pupils.

This brings us to the question (of very great importance in giving the teacher a proper estimate of the training value of following plans that she does not devise or originate) whether professional growth—growth in skill, in ideals with respect to methods, and in logical habits of mind—is possible in work in which there is little or no opportunity for originality. Would such work afford a measure of a teacher's power or possibilities?

e. GROWTH IN SKILL

There is a very strong parallel between the development of the machinist and that of the teacher. The apprentice learns his trade by repeating the experiences of the men who guide his training. Later, when he becomes a regular workman in the toolroom, the acquirements which as an apprentice he aspired to and which the employer is anxious to make use of and willing to pay for are skill and accuracy in the construction of that which is outlined for him in detail on the blueprint. Now these qualities are the results of a regular but imperceptible growth. They are developments from training and toolroom work in whose conception or outlining he had no part. So with the teacher. What we admire and value in her work is skill in execution. We forget to ask, we care little, whether she or some one else prepared the subject or the lesson blueprint. We think only of the finish and the logic of her presentations step by step and the quality of the completed product. It is the power that she displays in this work that brings her her recognition and reputation.

Skill in the schoolroom, as in the machine shop, is a development entirely from the proper execution of the work. The conditions and the only conditions of this growth are that the work follow logical principles and that it be properly organized. It is through the display of power to follow outlines not of her own preparation that we measure her skill. The teacher who is able to take up intelligently a detailed outline of work in which there is a more or less radical departure from the plans and methods with which her experience has had to do, give it its proper form, and bring out all the value that there is in it, displays skill of the highest order.

f. Ideals and Mental Habits

Ideals of orderly arrangement of teaching steps and orderly habits of thinking in matters relating to sequence and methods of presentation, are developed under the same conditions.

The teacher whose training and subsequent experience have been under definite plans in the application of definite principles which she has kept constantly in view, will develop habits of mind accordingly. She will feel a need for a proper arrangement of any original work that she may from time to time undertake. She will naturally attempt to find in such work the elementary teaching steps, their proper order, and the methods of presentation which will accord with the general plan upon which the work is based. She would naturally, in taking up a line of work in which she is not wholly familiar examine it at once with reference to the lines of order running through it and note whether this order has been consistently lived up to.

These habits of mind and these ideals of system and order are at the foundation of successful teaching and they can be acquired only by working in an atmosphere of this kind. The work itself affords the training in following outline work; the influences connected with the work develop the habits and the ideals.

3. *What Development Means*

Development is that form of teaching under which each step in the process is a simple natural broadening of the child's experiences in the direction of the subject with which at any stage the training has to do. Evolution applied to animal life relates to the progressive advancement under which, according to that theory, each step produces an animal of a higher type until the evolutionary process reaches its climax. Development is the evolution of the subject of instruction.

The theory in animal evolution is that there was in the development of each type a necessary sequence of environment conditions and that this sequence determined the direction, rapidity, and regularity of the development and the perfection of the product.

So in the development plan in the teaching of a given subject. The theory under which the term is used is that there is a sequence of teaching steps which if discovered will lead directly and logically to its practical mastery. This sequence changed, and the development ceases or is deflected from the desired direction, or the product becomes an indistinct or imperfect image of the subject.

Development ideals in teaching or development forms of work are not new. That the class work upon any given subject should be very carefully and thoughtfully planned by the teacher in advance is universally recognized and emphasized in the training of teachers. The principle insisted upon as underlying all plans is that teaching steps should be arranged in order with a view to the evolution, the systematic unfolding of the subject. It is under the influence of these ideals that development has become the *aim* more or less definitely in view in every seriously and intelligently considered analysis and organization of teaching units. The formulation, however, of a working outline under which development will be realized in the subjects outlined is the great problem in practical school work. The fact of a carefully prepared plan is not presumptive of a development product. The question is always in order. Is this a development? or, Is this an *approach* to that form of instruction?

4. *The Development Process*

Development, as has been stated, is the consistent, progressive unfolding of the subject to be taught. In view of the universal principle of the need of system and order in the organization of constructive work, it has to do very largely with sequences of unit operations—sequences of teaching steps.

Every subject from the teaching standpoint consists of a series of instruction elements, each representing a teaching unit. It is a fundamental principle in school work that in any new step in the teaching of a subject there may be but one unknown factor and this an *element*. It is in view of this law that development requirements involve the analysis of the subject to be taught into its elements in order that the separate teaching units may be known.

The teacher's working plan must involve the arrangement of these teaching steps, these elements, in the order that will contribute to the development purpose—the regular, unbroken growth of the subject complete and clear. The organization of the sequence in any subject is a very difficult problem, the most difficult that the teacher has to deal with. An outline that from the standpoint of the mature mind seems faultless in its pedagogy may prove to be utterly inadequate and useless when put to the test in the schoolroom. The teacher who undertakes such an outline must have qualities of persistence and open-minded patience.

5. *Development Must Have Proper Foundations*

Every development, to whatever branch of work it may relate—number, geography, science, or other subject—is dependent upon some form or forms of previous experiences through which the pupil may interpret the new experiences that the development work will involve. Some of these are required in the early steps of the development, others in later stages, but all must be ready in advance in order that the development process may be unbroken in its continuity. A child of seven or eight years of age, for illustration, has had experiences ordinarily that would enable him to understand the part of introductory fraction development that relates to the terms fraction, unit, integral, number, fractional number, mixed number, etc.; but if he is unfamiliar with the number processes, objective and written, the operations of simple addition, subtraction, multiplication, partition, and division with integers, he is totally lacking in the experiences which are indispensable for the interpretation of the steps as they begin to relate to these operations with fractional numbers. Without this previous preparation the work halts and the continuity of progress is broken while the child is taken back to the development or review of these lines of experiences upon which fractions rest.

The teacher who enters upon a development without first a careful survey of the situation to assure herself that all the experiences, objective and otherwise, requisite for proper interpretation at all stages of the work are ready and fresh jeopardizes the success of her development plan. The suspension of work to prepare the class for that which will be necessary for interpretation at any stage breaks the continuity of the subject growth and cools the enthusiasm which the connected series of progressive experiences arouses.

It is not necessary that a given development process constitute the only line of number work in a teacher's daily program. The taking up of other unconnected lines to be conducted as parallels to it is not a break in the sense that it would interfere with the line of thought. In fact there should not be just the one line of number work during the course of a development. Each subject (see *B*, *6*) should have parallel work on reviews or "minor" subjects. It is the deviation at a point where thought should be continuous to something that is practically a part of the line under development, a preliminary to it, that produces the disturbance and is harmful.

This matter of attention in advance to the preliminary experiences necessary for an unbroken development is of utmost importance, therefore, in the organization of development material.

6. Classes of Development Subjects

a. The subjects to which the development plan of instruction may be applied may be grouped under two general classes:—

(1) Those in which the instruction has to do with *form*—the form that a natural growth gave, the changes in form in each case through which its evolution carried it. In these sub-

jects the developments follow the lines along which they grew historically.

(2) Those in which the instruction relates less to form and more to definition or description and to the nature and the relations of the processes. In these subjects the historical changes were slight or without pedagogical significance.

Number furnishes in decimal fraction *notation* and in common fractions *throughout* two strong illustrations of those two classes of subjects.

NOTE.—The only differences, practically, between common and decimal fractions is in notation and in the fact that in one the denominations are unlimited and in the other limited. In all other respects—in the nature of the fractions, in the manner of oral expression, in the significance of the fundamental operations, and in the principles involved, the two must be classified together.

b. SUBJECTS OF THE SECOND CLASS ILLUSTRATED

In the subject of common fractions the emphasis points are those which relate to the nature of fractional number, its relation to integral numbers with which the pupil's previous experiences have had to do, the method of expressing fractions in written form, and constructions which bring out the fact that each of the fundamental processes with fractions has the same effect upon quantity as the corresponding process with integral numbers. "Form" as a problem in common fraction development is of minor importance. Common fractions were not, in form of written expression, the result of a natural growth, the product of the operation of a law. This history while interesting had no development significance. They were given their form arbitrarily, this arbitrary establishment coming into general acceptance and use because of its simplicity and directness. The subject is almost purely, therefore, one in which we deal with *definitions and relations*.

c. SUBJECTS OF THE FIRST CLASS ILLUSTRATED

Decimal fraction notation, on the other hand, is the product of an evolutionary process that began in the childhood of the race and may be traced in unbroken course to its culmination three centuries ago in our full decimal system. The proper plan of instruction is through the study of this history and the dramatization of the methods of expressing number and of computations in the successive stages of its growth. The subject is therefore one of *form*, one in which there is an historical order that the instruction plan must closely follow.

7. Development Work and Orderly Mental Habits

The effect of this plan of conducting class work in each subject on the basis of the pedagogical sequence of the successive teaching steps, is to bring each element as the development proceeds into relations with all the experiences that are needed for its full and clear interpretation. It makes the teaching of a subject a building up process, a construction, in which, upon a foundation of previous experiences, the structure is made to grow naturally and regularly toward a complete and symmetrical whole. At every stage in the teaching process the new element has in this way a logical connection with the sum of previous experiences in the subject—a connection which the child learns to expect, to look for. This regular unfolding of the subject has a strong influence upon mental tendencies. It makes for the development of habits of seeking possible relations, of thinking in an orderly manner.

The logical mind is the mind that habitually seeks relations. This habit of orderly thinking is a product of school and other training agencies. If, running through the work in every subject of instruction, there is a constant natural succession of effects from causes, the educational process becomes one of tracing relations. The child's mind is made to act in an orderly manner, to acquire the habit of seeking for effects from causes and causes from effects. In this way the plan contributes to the habits, the tendencies, from which reasoning power springs.

The influence of development plans of teaching is not limited to habits of thought and reasoning power. Memory also profits in the same manner and to the same degree. We remember material in the form in which it comes to us. If it is acquired under a teaching plan in which logical sequences are not considered we remember it in unrelated form. If it is acquired through reasoning processes, it is in related form in memory. Memory's hold upon unrelated knowledge is uncertain. On the other hand there is a tendency for memory to lay hold of and to retain material that is acquired through the exercise of reason. Where we give to the student, therefore, material in organized form, we give him that to which there is a natural tendency of the memory to cling. This influences memory habits and power.

Development plans of work, therefore, make for orderly habits of thought and memory, power to acquire truth by reasoning processes, and power and a tendency to hold in memory that which has been acquired.

8. Elements Classified as Fact and Corollary

a. CLASSES DEFINED

The elements into which a subject is subdivided for teaching (development) purposes may be classified with respect to their relations to each other as independent or fact and dependent or corollary elements. An independent element is one that is in the nature of an arbitrary *fact*—a fact not discoverable by the pupil by reasoning. Such an element, in the teaching process, must be *given* to the child—given as an item of information that is not within his power to acquire unaided. A dependent element is one that in its proper place in the teaching order would be a corollary to a "fact" element or to one or more other corollaries and would be discovered by the pupil by inference.

b. ELEMENTS THAT ARE ABSOLUTELY "FACT"

This classification, however, is not absolute in that a given element is either independent under all conditions or under all conditions corollary. The position that is given to a dependent element may entirely change its character in this respect and consequently its classification. There are elements in every subject that are essentially independent in character regardless of conditions of position or sequence. As illustrations, we have—

The term *fraction*;

The *form* of the written fraction, numerator above and denominator below the line;

The *cost* as *whole* or *base* in loss and gain;

The *marked price* as *whole* in trade discount;

The *amount due* at maturity as the *basis* of bank discount; etc.

c. ELEMENTS THAT ARE CONDITIONALLY COROLLARY

The classification of an element, other than one that is necessarily an arbitrary fact, is dependent upon the position given it in the teaching order. A dependent element, a corollary, is an inference from an accepted truth. Logically, therefore, its position in a teaching sequence must be immediately *following* the truth out of which it grows as an inference. If it should be brought under consideration in class work in *advance* of its basal truth instead of directly *after* it, it would not be inferable.

Its character instead of being that of corollary would be that of an independent unit of instructions.

If, for illustration, x, y, and z were three elements of a given subject in their proper sequence, x an independent and y and z elements inferable from x, y and z would be corollaries. If, however, the instructor in organizing his work should through error in judgment arrange these units in teaching order so as to make the sequence z x y, z would change its character from corollary to "*fact*" because *preceding* its basal truth and there would be in place of the fact—corollary—corollary order (xyz) a fact—fact—corollary succession (zxy). If he should organize the elements in zyx order, there would be a succession of *three* "facts" both of the original (xyz) dependent elements assuming, because of their places in the teaching order, the character of independent units.

d. THE TWO CLASSES COMPARED AS TO THEIR PRODUCTS

It has already been shown (see 7, page 17) that school work when conducted with strict regard for analysis and the sequence of teaching steps makes for the development of orderly mental habits, reasoning power, and memory.

All subjects under development plans influence each of these qualities but not in equal degrees.

Subjects differ, however, in the extent to which they influence these faculties. Subjects which we have classified as "form" subjects contribute particularly to mental habits; and subjects which have to do with "definition and relations" contribute more to reason and memory.

This difference in training product is easily accounted for by the fact that the two classes of subjects differ in the proportions of their units that are strictly "dependent." Historical subjects like the subject of decimal fractions have strong sequences but there is a comparative poverty of *corollary* elements. "Definition and relation" subjects have looser sequences but are rich in the proportion of the elements in each that are possible corollaries. It is to this larger proportion of these corollary elements that their higher training value in reasoning power is due.

e. How This Classification Affects Organization

What the school work does for the pupil in the way of the development of proper mental habits and power to remember and to apply may be fairly judged by the extent to which the instruction preserves the nature of corollary units in each subject.

The value of a teaching plan must be measured, therefore, by the discrimination with which possible dependent elements, through giving them their proper place in the teaching order, are made to retain their corollary character. Every dependent element that is made independent by position lowers this value in that, its corollary nature being lost, it compels the teacher to *give* to the pupil a truth that the latter should discover for himself through the exercise of his reason.

The importance of this classification of the elements in a subject, therefore, lies in its ultimate effect upon the development process. An error in the classification results in an error in the organization of the class work and a consequent lowering of the quality of the development product. To judge of the value of a development plan one must examine it with reference to the discrimination that has been exercised—

(1) In this classification of the elements; and,
(2) In determining the order of the teaching steps.

If the classification of the elements is defective, the plan is defective. Under such conditions the teaching order must be defective. If the *classification* is good, there will be *elements* of strength, at least, in the *teaching order.* The power to classify elements carries with it a degree of power to arrange them in proper order.

9. Importance of the Sequence Order

The aim in all working plans for class work in subjects to be taught in our modern education is to conduct the work in a way to give the child the power and every possible opportunity to discover the truth involved in a development element, when he is brought face to face with it, by the exercise of his reason—by inferences from his own experiences.

The extent to which inferential reasoning may be used in the development of a subject depends almost entirely, as we have seen, upon the teacher's arrangement of the material. Every error in judgment in this organization work lowers the teaching value of a plan. This order of steps is therefore of utmost importance that no possible corollary element be relegated to the "independent" class and an opportunity for the child to discover truth by his own reasoning be lost.

B. Principles in Teaching Practice

1. Major and Minor Subjects in Number

The various subjects in arithmetic may be divided as to the need of using them for regular or for incidental instruction into two classes which may be described as major subjects and minor subjects.

a. Major Subjects

A major subject is one that is more or less complex and whose development on this account must be given for a considerable period of time—several weeks or several months—the important place in the daily number program. It is a subject that has a variety of forms and processes. These make necessary the development of special imagery and special language, work upon which must be continuous as well as systematic. Among these subjects are common fractions, decimal fractions, percentage, etc. In each of them there is a logical connection of parts which the teaching process must recognize. Each is rich in corollary elements. Without regular, continuous development the logical connections do not manifest themselves, corollaries which the child should discover become facts which he must memorize, and the whole purpose of the development is defeated.

b. Minor Subjects

A minor subject is distinguished from a major, in general, by the way it lends itself, on account of the nature and relations of its elements, to the development plan of instruction.

In the major subjects there are elements that are logically interconnected. Some of these are independent and others corollary, thus lending themselves to a development plan. The minor subject has little and in some cases nothing that admits of development.

In the major subject the operations are various and closely interrelated. In the minor (excepting in the case of denominate numbers, as we shall see) they are limited to a single process.

Unlike the major subject, there is in the minor no series of parts or elements which the teacher for instruction purposes must arrange in pedagogical order and therefore no corollaries for the pupil to discover and work out. There are, however, teaching steps that must have their proper order, but such order is based upon degrees of difficulty and not upon *logical* sequences.

Among the minor subjects are long division and the other fundamental operations with integers, denominate numbers (exclusive of surface and cubic measures), and simple interest. These are subjects in which there are no elements or principles or operations that are outgrowths of others. In the terms or other facts which the child should learn in any of these subjects, there is no necessary order of approach, no logical order in which they must be taken. The measuring units in denominate numbers, for illustration, were established arbitrarily. The quantitative value of any one of the units in a given table has no connection with the value of the higher or the lower unit in the same table. There is no logical significance in the fact that there are three and not four feet in a yard, in the fact that the number of pints in a quart is two, in the fact that the gallon has four quarts, or in the fact that an avoirdupois pound is subdivided into sixteen definite units.

There is no loss in the way of continuity or opportunity for inferential reasoning in teaching during one school year parts of certain tables; during another year, other parts of these tables; and later still, other parts. There is no reason in pedagogy why the pupil who has learned ounces, that there are 16 ounces in a pound, should or should not be given as the next item of instruction that a ton of coal weighs 2000 pounds. There would be no illogic, no violation of sequence, in the reverse order—in teaching 2000 pounds in a ton of coal or hay before the pupil has become familiar with ounces and knows that there are 16 ounces in a pound. The term ton is a familiar word. It is as common as ounce, perhaps it means more in the child's experience.

2. *The Operations Compared as to Corollary Elements*

In the major subjects the operations are in a greater or less degree complex and interconnected. They must be taught, unfolded, step by step. It is with these operations, largely, that the development plans in such subjects have to do. There are operation elements that must be discovered and arranged in teaching order. The major subjects are major because of these constituent elements and their interconnection and because of the need of the adoption of a plan of work with respect to the operations that will give unity to the teaching product.

In the minor subjects, on the other hand, the operations are simple, there are no elements to be arranged. In long division, for illustration, there is no variety, it has no sequences. The pupil who knows well the so-called "facts of number" and has learned the form that the operation should take (and this applies in each of the other fundamental operations with integers) has acquired all that the subject has for him in the way of information. What he needs further is practice—practice for accuracy and rapidity.

The same may be said of simple interest. The pupil who has found the interest on $4 for 3 years at 2 per cent. (and such solutions are very elementary, requiring no instruction or even guidance except as to the proper arrangement of the written work) has the only operation that simple interest solutions require. There are teaching steps in the subject—finding interest for years as in this example; very much later (months later) finding interest for years and months; and months or years later, finding interest for periods which involve days—but these steps are arranged in an order that is based on the difficulty of the operations and not on considerations of logical sequences. Here too the child's problem is practice—practice for accurate and rapid computation.

The operations in denominate numbers (excepting surface and cubic measures) are also simple and single. The child in the primary grade who has reasoned out the reduction of gallons to pints has used the only process that there is in reduction descending; and when he has reasoned out the change of feet to yards or of pints to gallons, he has used the only reduction ascending process that any solution will require. Here, too, in each of the two forms of reductions there must be an arrangement of the teaching steps on the basis of the difficulty or complexity of the operations, but the character or form of the operation does not change. Here, too, except for the denominate number facts that he must learn to make use of in a practical way and must memorize, the child's problem is practice—practice for accurate and rapid work.

3. Definitions and Oral Statements of Processes (Rules)

A very important part of the class work in any subject in arithmetic is that in which the pupil is required to give orally or in written language statements of how he made this or that construction, what a given process consists of, and definitions of the terms that he has learned to use. Work of this kind is absolutely essential to clear and complete imagery in subjects that have been developed, no matter how carefully the work has been planned and conducted. It is the only means of clearing up the points of emphasis in terms and processes—points that the pupil has not observed closely enough to leave an impression on the image. The statement of a process, to describe it completely and in good English, requires the full concentration of the attention upon the process. The pupil may be able to make a construction, to perform an operation, or to recognize the meaning of a term; but when he is asked for the first time to describe the construction act, to define the term, or to state the nature of the process, the incompleteness of his description will show that there are important features of the process or of the term that his mind has not been centered upon, that are practically not a part of his image.

He may be able to make a half or a fourth or a third, for illustration, and to recognize it when he has properly tested its value; but there will be points of careless observation, failure to note all of the essentials of the construction, until statements of the construction process forces it upon his attention. It is a matter of almost unbroken experience that the child's reply to the question, How do we make halves? is, We tear the whole into two parts. The essential, the *emphatic* feature of the construction, that the parts must be *equal*, is not in his statement and, of course, not strongly in his conception of half. This, the oral statement after it has been developed clears up. It puts the fact of the *equality* of the *parts* into the center of the image where it should be.

Not only does this oral work, the defining of terms and the stating of processes, make for clearer imagery but it is a distinct aid to memory. Such a result would be inevitable. An image that is clear is naturally more enduring than one that is dull or imperfect.

The stating of a process in oral or written language makes the pupil more sure of himself, more confident in its use. He may, for illustration, be able to find the area of a triangle without having ever stated the process; but he will remember it very much longer and be far more confident in computing such areas if he should be required to put the operation into spoken language—Multiply the base by the altitude and take half of the product. The old rule for the circle committed to memory and recited time and time again, Multiply the square of the radius by 3.1416 or Multiply the square of the diameter by .7854 or Multiply the circumference by the radius and take half of the product, has been the sole, confident reliance of many a man in circle measurements —a process that he would long ago have forgotten but for this rule that lingered in his memory.

It is a part of our common experience that we know best and remember longest that which we have many times stated and restated orally or in writing. The individual who has formulated and many times repeated a rule for taking the square root of numbers knows square root better and will remember it longer than another person of equal or even greater experience in square root solutions who has not had the opportunity to recite orally or in writing upon the process. If we wish a child to have clear imagery and to have a given fact or principle or process deeply impressed upon him, we must give him the opportunity to recite upon it frequently in oral and in written form. The strength of the arithmetic of a generation ago was, in part, its insistence upon oral definitions and statements of processes (rules). The arithmetic of to-day makes its appeal to reason, and to memory based on reason, but the full advantage of this plan of teaching can be realized only by the retention of the oral features that characterized the old.

4. Rest Exercises and Concentration of Thought

A change in work is one of the ways of resting and refreshment in mental as well as in physical work. A long succession of exercises at one time on one subject—operations, for illustration, under the fundamental processes with integers, under common fractions, under interest, or under other subject—very soon reaches the point where it becomes drudgery. The effect, under such conditions, is that the rate at which the individual works tends to slow down and accuracy suffers because of diminished power to hold the attention upon the computations.

If, instead of this uniformity in the operations, two or three of the exercises should be replaced by as many review exercises in other subjects—one in common fractions, perhaps; one in denominate numbers; and one in simple interest—the changes would mean relief from the monotony of the regular work; rest; new thought associations; and the full concentration of thought that is necessary and possible when the mind turns to a new line of computations with which it is familiar. If the regular work thus broken into for the two or three rest-review exercises is decimal fractions, it would make it necessary for the pupil to drop the decimal fraction line upon which his mind had been running and make a new concentration—this time upon common fractions. This concentration would then be dismissed and that upon the units and operations and written forms in denominate numbers might take its place. A simple interest exercise following this would center the mind upon that subject. The changes make full concentrations necessary and natural, to say nothing of the fact that the concentrations are upon subjects which must be kept fresh by constant reviews.

The child after this digression comes back to the remainder of the regular exercises, fresh and, under the stimulus that the changes in his work have effected, able to work more rapidly and to concentrate more fully. The loss in the number of exercises in the regular subject by the substitution of the exercises in the other subjects is more than made up through the gain in the power to concentrate—that for which, in large measure, exercises in computations are given.

The tendency of such organized variety in the number written work, therefore, is to keep the pupil's mind fresh and alert during a working period and in condition to center his full attention upon his computations. It affords the opportunity for reviews, furnishes conditions of work that develop power of concentration, and makes for accuracy.

5. *Reviews*

All teachers realize in a practical way the need of reviewing the work of the term or the year in every subject in which they have given instruction; but the extent to which general reviews should enter into the program of the elementary school—reviews of subjects that other grades have given—and the responsibility of the teacher in the matter of such reviews, are not sufficiently recognized.

There is no time in the child's school life where reviews of previous work can be discontinued without there being losses from forgetting the old to offset partially, at least, the gains from the study of the new. A subject that is permitted to fall into disuse is soon to a great extent forgotten. This is true in the cases of children in school, it is true of people in the ordinary affairs of life, it is true of teachers. The principle applies in arithmetic work, in geography, in history, in algebra—in all subjects. We remember the geography of a given part of the world if our reading or our special interests in any form lead us to use the map of that region. If we cease to use geography, we forget it. If we are readers of history constantly, or if special interests keep it before us, we retain what the school gave. If not, whether in this subject we are laymen or teachers or students, our history acquirements shrink to a shadow of what they were at the close of the school course in the subject. We have no more reason to expect a pupil who has learned common fractions, even under the most approved plan of instruction, to remember their processes indefinitely when he is given no occasion to make use of them than to expect us older persons to remember our square root or cube root or bank discount or the linear or square or cubic tables of the metric system after a year or two of non-use of them.

The purpose in teaching number is to give the pupil the tools for measurements. The subjects which form the indispensable tools for almost all forms of activity in which measurements are necessary even to the simplest forms of business are the "number facts," the fundamental operations with integers, common fractions, denominate numbers, and decimal fractions. It is hard to conceive of a form of activity, above the level of the commonest labor, in which these subjects are not a part of the workman's necessary equipment. The tool is of value to the individual as long as he is able to use it. If at any time for any reason he loses the power to use it, he has lost the tool. The pupil, for illustration, may learn to compute with common fractions. He has the common fraction tool. Afterwards through having no occasion to use them, he forgets all but the shadow of what he learned in the subject. The tool is lost.

Applying this principle to schoolroom work, let us take the case of the teacher whose "major" subject in number for the term or the

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year is decimal fractions. Her assigned task is to add this to the child's available tools in the subject. If, when this task has been accomplished, the child is found to have lost the power to compute with common fractions, the decimal fraction teacher has added nothing to his tools in number. She has merely substituted the decimal fraction tool in place of the common fraction tool. The child is little richer. There is little to the teacher's credit, as the result of the term's work, in the way of added number power in the child.

The teacher's responsibility for reviews, therefore, does not begin and end with the subject or subjects which she is called upon to develop. Her responsibility is cumulative. If the major subject is common fractions, she cannot avoid being accountable for "number facts," for work toward accuracy in fundamental operations with whole numbers, and for such parts of denominate numbers as have in previous years or grades been introduced. If her major subject is decimal fractions, she is responsible for the "number facts"; for exercises for the development of accuracy in the fundamental processes; for parts, at least, of denominate numbers; and for operations with common fractions.

The organization of regular and discriminating reviews of the essentials of work done, in order that the results be not lost through non-use, is as necessary to school efficiency as the organization of the advance work; and the recognition of the need of such work and of her own responsibility in providing for or executing it is an important product of the teacher's training. If all of the elementary school work for the children of a community is in charge of a single teacher, she is the organizing, the responsible agent. If the work is in the care of a group of teachers, the organization is the work of the supervising authority. Without such regular system in reviews the school organization is lax, time and energy are wasted, and the product is retarded number growth.

6. Parallel Lines of Work

a. THEIR NATURE AND USE

By a parallel in number we mean a subject that is under development or under review in connection with the development of some "major" subject. We have seen that in the interests of the development of power to hold the full attention upon the operations in seat written exercises every major subject must have with it one or more other subjects conducted as parallels, sharing with it the time allotted to number on the daily program. This parallel or these parallels (there may be one or more in use at one time) are the subject or subjects represented by the problems that are given for rest and change—problems that afford relief from the monotony of repeated exercises on one subject or rest from fatigue that application to unchanging work induces. Computations in number, especially those given when accuracy is the special aim, must, as we have seen, have the variety, the change, that parallels afford. Fractions, for illustration, is a major subject. As such, it is the principal feature of the daily work in number during a large part of a year of the elementary school course. Without parallel lines of work it would be the sole computation subject in the daily seat number period. As such it would be without the stimulus that there is in some degree of variety, without the change in work that induces concentration. Parallels, because they induce concentration, shorten the time that is required to develop accuracy in fraction computations and on that account favor fraction achievement. The pupil makes his fraction solutions under conditions of closer attention and clearer mind and thus gains rather than loses in the subject notwithstanding the shortening of the time given to it because of the introduction of the parallels.

b. Selected Without Regard to Sequence

A parallel problem—a problem from a subject that is essentially parallel or that is used at the time for parallel work—is, as we have seen, one that is incorporated in a series of seat computation exercises on some major subject to give variety to them and thereby make concentration upon the computation work possible.

Only one "major" subject may be on the class program at one time. If, for illustration, the teacher is developing decimals, she could not in the same period of time, conduct, as a parallel, a development of cubic measure without halting the former. Such a temporary suspension of the decimal fraction work would be possible and not at all objectionable provided the suspension is made at some climax point in the subject—after that part of the development which has to do with reading and writing decimals, after addition, after subtraction, or after multiplication, etc. A halt might be made at one of such points, if thought desirable, for the development of another subject, provided it is comparatively short—cubic measure, for illustration. In that case, the work that has been completed in decimals must be subject to constant reviews in the meantime as a cubic measure parallel. To carry both of these subjects as parallel *developments*, however, would "slow down" the work in each so much that interest could not be maintained in either.

Parallel problems, therefore, can not be from a subject or a part of a subject that is under development. They must necessarily be problems that we may speak of as detached, incidental, not a part of a continuous line. In other words, they can not be problems that must be selected upon considerations of order or development sequence. Each must be regarded as an independent unit, a practice problem complete in itself.

c. REVIEWS AND "MINOR" SUBJECTS ARE "PARALLEL" MATERIAL

Reviews and minor subjects lend themselves to uses as parallels. Exercises in which there are no major subject solutions, are naturally, almost necessarily, made up of detached, incidental problems.

Reviews of decimal fractions may be taken as an illustration. It is recognized that such work should be incidental. The review is stronger for using scattered exercises—exercises that involve, perhaps, first a division solution, then one in addition, then one in multiplication, then one in writing fractional numbers, etc. As a general rule the less there is of logical interconnection in the series of problems that make up a review exercise the more valuable it is as a review.

"Minor" subjects are by definition subjects in which a logical interconnection of the teaching elements is lacking. The computations in these subjects represent single, simple operations. There are differences in computation difficulties, however, due to the quantities involved. In some, the solutions may be made mentally because the numbers involved are small. Others are more difficult because of the use of larger integral numbers or fractions or because of conditions which make fractions necessary, as in interest for years and months or for years, months, and days. In the selections of practice or applied problems in minor subjects, therefore, there must be considerations of grade (the grade of the problems rising as the pupils gain in power to operate with higher numbers and with fractions) but not considerations of sequence. The problems in "minor" subjects, therefore—denominate numbers (excepting square and cubic measures), long division and the other fundamental operations with integers, and interest—are incidental, out of logical relations with other problems in the same subject. They may be used in class work (whether as parallels or as complete exercises) in unrelated units—a solution at one time in a long measure reduction, at another time in an ascending reduction in liquid measure, at another time in an exercise involving long division, at another time in ascending or descending avoirdupois weight, at another time in interest. In fact, in whichever use they occur they are from their nature individual, a series of detached exercises. It is impossible to change their character in this respect. They are therefore naturally and always proper parallel material.

d. REVIEWS AND MINOR SUBJECTS ENTIRELY THROUGH PARALLEL USES

There are two ways in which instruction or training (under whatever name we describe it) in minor subjects and regular reviews of subjects that have been developed may be conducted:—

(1) As independent instruction subjects, a regular portion of each year or term being set aside for each; or,

(2) As parallels to major subjects, no separate periods of time being allotted to them.

Under either plan the classwork aims are realized entirely through computation exercises. As to instruction in computations, none is necessary. It has been shown that minor subjects in the simple forms of their operations require none. The pupil is independent of it because it involves in each subject a line of reasoning and an operation with which he is familiar through the ordinary mental applied problem work that in every properly organized course of study parallels work in "number facts." In the very simple forms of solutions, the pupil's knowledge of the significance of the fundamental operations—the effect of each upon quantity—is sufficient for the operations. Neither in these nor in those of the higher grades of difficulty—higher because of higher integral numbers or fractions—will the pupil require instruction or assistance if his work is properly directed. He will make the computations of each succeeding grade independently of assistance, excepting as to arrangement of written work, provided the work is so organized for him that it carries him regularly from solutions of one grade of difficulty to those of a next higher. Under either plan the teaching problem would be one of directed practice. The product would be the same whichever plan is followed: the development subjects kept fresh by reviews and the minor subjects acquired.

The choice between these two plans is a matter of very great importance in the organization of number teaching because they differ materially in cost—in teacher-hour and child-hour cost. Under the former, the work of reviews and minor subjects would require that time be specially allotted to them—time to be used for instruction and for computations. Under the latter, such work conducted as parallels would be done in time allotted to the major subject.

It has been shown that parallels stimulate and expedite the work (see *6 a*, page 39) in the major subjects, that they hasten the development of power of sustained attention, one of the important aims in number training. Under such conditions the gain to the major subject—the real gain not only in quality of product but in actual time would more than make up for the part of each computation period given to the parallels.

e. The Time Element in Parallels—Time for Power to Develop

The element of time—time for the pupil to develop power to take hold of the successive grades of the work in a minor subject from the first simple computations to those of the more difficult grades—should be understood by the teacher. The actual time that must elapse—time measured in months and perhaps years between the first mental computation in a subject and its completion according to elementary school standards—is by no means short. Denominate number work, for illustration, must be extended over years of the elementary school course. There are forms of measuring that enter into the practical life of the youngest pupils and the school should give them opportunity to become familiar with the units and to make use of them in measurements. It is one of the ways in which educational work should touch the child's every-day experiences and make use of them. The reduction (oral) of quarts to pints and of pints to quarts or gallons is a part of early primary number training. Such work goes with the training in the "number facts" as practical applications. Reductions oral and written under dry measure, under long measure, under some parts of time measure, under avoirdupois weight, and under some of the miscellaneous measures should be parts of the early number training—should all find places as parallels to the regular development subjects of that early period.

The same principle applies in simple interest, another minor subject. It should be introduced at the age when, in the opinion of those in charge of the course of study, the pupil may profitably take up the work under common fraction development. It should be used as one of the parallels of common fractions, of surface measure, of cubic measure, of decimal fractions, of percentage. From the time when it is first introduced it should be in regular use in the lower, then in a higher, and again in a still higher grade of operations until it is completed years later in the grammar school period.

f. CONSISTENCY AND UNIFORMITY IN ARRANGEMENT OF SOLUTIONS

It has been shown that in the minor subjects no teaching* is necessary except as to the form in which the written work should be arranged.

This matter of the arrangement of the written work in minor subjects, in denominate numbers, and in simple interest especially, should have careful attention. Each should have an order in which its analytical statements should be arranged. This form or order of arrangement should be simple and not changeable. There should be, for illustration, but one form of arranging the statements in reduction descending operations in all of the denominate number subjects—the form that gives each statement in the analysis the place with reference to the other statements that orderly thinking requires. This is entirely practicable because logical order is the same under reductions of one kind of denominate numbers as under those of another kind. There is every reason, in the interest of training to orderly thinking, why there should be this uniformity of arrangement. There is no reason why a definite order should not be insisted upon in the computations. It is a part of the child's training in logic.

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* This must be understood to refer to the *operations* only. Instruction is necessary in the denominate number tables, in the character of the measuring units in each, and in the method of applying these units in practical life. Such instruction should be most carefully planned and given.

7. *Acceleration in Number*

The broadening of the elementary school curriculum without lengthening the working day or the year places upon the schools a problem similar to that which industrial competition has been forcing upon the shop, that of a higher product per workman-hour.

For the schools it is a problem of wise and safe acceleration. They must give to the child greater reading power in his first year in school and in each succeeding year—greater power to understand the written page and greater power to read expressively. They must give him a broader and more practical knowledge of geography; more power in oral expression; greater power to put thought into written expression, so that what the child writes expresses that thought fully and clearly. These with music, drawing, history, and civics, hygiene, spelling, writing, handwork, the science of common things, etc., compel the most careful use of school time and wise expenditure of school energies.

When we use the term acceleration we must use it as applying to the hastening of work in general. Applied to an individual subject it is a dangerous word. We may, for illustration, hasten general progress in arithmetic, but the hastening of work in common fractions or decimals or long division or percentage has possibilities of imperfect digestion and assimilation. To "speed up" any subject beyond the pupil's power to grasp and assimilate it is not acceleration in any proper sense of the term. Such work would be wasteful. It would retard instead of hasten. This must not be taken to mean that assimilation may not be expedited. Such is not the case. It is a matter that does not admit of discussion that one method of teaching a subject may make assimilation more rapid as well as more perfect than another but that under a given method applied in a way to give it its best results acceleration might prove to be destructive. There is no way in which judgment and conservatism is more necessary than in this matter of rate of progress in number subjects.

Acceleration in the shop is not attempted by making the tool cut the steel faster or by increasing the rate at which the great wheel in the engine room revolves. It is secured in the plans and tools of manufacture so that one machine will do the work formerly requiring two; by placing and adjusting machines and tools to provide against delays in successive operations upon a given article by one machine or tool and then another; and by other forms of organization to save time that would be lost through steps and motions that count for nothing in the actual manufacturing process—organizations that have for their aim to put, so far as possible, full sixty minutes of productive work into every hour of the workman's time.

This is the form of acceleration that should apply in number—the form that reduces waste in time by every legitimate means.

We have seen that there are subjects in number that may be used as parallels, that these subjects under parallel uses save the time that would be given to instruction in them were they used independently, and that parallels improve and hasten the work in the development subjects with which they are connected. It has been shown that denominate numbers (excepting surface and cubic measures) are natural parallel material and may be taught wholly as parallels to other subjects if the work is begun, as it should be, in the early number period and continued through the elementary school; that when so used they save entirely the time that must otherwise be given to them; and that, in addition to this, their use expedites and improves the quality of the work in the subject or subjects which they parallel.

It has been shown that simple interest is another parallel subject; that it may be used in the same way as an exercise for rest and change· that when introduced early in the course in number and used as a parallel, it may receive its full development thus incidentally in connection with work in major subjects; and that such use not only requires no special time to be allotted it but shortens the development time in the subjects with which it is used.

Here is the real field for conservation—conservation through organization of the work. It suggests to the teacher how one subject, a development subject, through its need of parallel exercises for rest and change, may be made to carry entirely the burden of a strong minor subject without loss in quantity or quality of product and thus in terms of the shop make one machine do the work and give the product that under other conditions would require two. This is constructive acceleration. It not only materially shortens the time that is required to complete the course in number but preserves and improves the quality of the work. It has been shown that reviews also are natural parallel material. Through their regular, systematic use as such they contribute in this same indirect way to practical acceleration in that they keep fresh the teaching product in every subject that has been developed so that no work that has been completed during the term or year or in previous years is permitted to become dimmed or lost through disuse.

8. Rapidity in Computation

Rapidity in the fundamental operations is relative. Individuals differ in the rates of their mental processes and therefore in the rates at which they can compute, just as they differ in this respect in their physical motions. The fact, however, that one child thinks or expresses thought or solves a problem more slowly than his neighbor affords no presumption that his mental power is of a lower order. The reverse may be found to be the case forty years later after the two have taken their places in the practical affairs of life.

Each pupil has his degree of rapidity with which he is able to work in arithmetical operations. Some would be rated as slow, others as rapid. This rate of work may be increased by wise stimulation. It is delicate ground, however, upon which to tread. It should be attempted, but attempted with great care because of the danger of over-stimulation. When we secure rapidity at the expense of accurate results the product is, to say the least, of doubtful value. The employer desires rapidity in computation. He is willing to pay for it. When, however, rapidity is secured at the expense of dependable results its value is lost.

9. *Accuracy in Computation*

When the service involves computation, the employer must have results upon which he can rely. Without it, service has absolutely no value. He will pay for accuracy, however, even if computations are made slowly. For rapidity with doubtful accuracy, there is no market.

Accuracy in any line of operations is a quality that is dependent upon power to concentrate the thought closely and steadily upon the work. This power is a development. It is a quality that the individual acquires gradually through the regular, systematic, continued practice in computation under conditions of full concentration of thought upon the operation. Practice in which the mind is thus fully concentrated makes for the development of the power to concentrate steadily and of accuracy. Practice in which the attention is intermittent defeats its own purpose. Practice in long division, for illustration, in which the thought is fully absorbed in the process makes for the development of this power and for uniform accuracy of long division results. Computation, however, in which the thought is wavering counts for nothing toward power of sustained attention or of computing with dependable results.

Concentration upon the operation therefore, is a prerequisite to accuracy. For the individual who knows well the "number facts" (there can be no accuracy, otherwise) the test of this power of sustained attention in computations that are practically only fundamental operations is uniform accuracy of results.

The organization, therefore, of our work in number instruction which as we have seen effects the acceleration of general work in that branch and the development of power to concentrate fully upon our computations gives us as a by-product the very necessary power of dependable results.

10. Some of the Qualities Essential to Skill in Teaching

Skill may be described as a combination of those qualities which make for successful leadership in the schoolroom—the teacher being the leader under whom class activities are conducted. Some of the qualities should be noted here, especially those which have a particular bearing upon development work.

a. The teacher who knows the value of reviews in construction and oral work and uses them wisely to hold on to that which has been developed so that nothing is lost has one of the qualities included under this term. Failure to keep alive the essential features of a step that has been developed so that it is ready for use at any time without the necessity of even a partial re-development shows a lack in this regard. The rule must be regarded as absolute that, if a step in a line of sequence is worth while as a step, it is worth preserving by reviews.

b. The teacher who knows when she may safely take up a next step in a development so that mental digestion will proceed without interruption, has another of these necessary qualities. To state this matter negatively, failure to note the point when a new step may be introduced with safety evidences a lack of that which is essential to skillful leadership. Progress may be too slow for interest and the development of proper mental activity or it may be too rapid for assimilation. There is the middle, the safe rate of progress in each subject, the determination of which requires skill of a high order. This rate of progress in any development is a matter of judgment with each class. Progress cannot be gauged by the calendar or the clock.

c. The teacher who fails to put energy and vitality into her schoolroom leadership cannot hope for a high rating in skill. The best arrangement of teaching steps with the best outline of presentation details is hopeless in such hands. These may be classed as qualities of skill because they affect the teaching product so radically.

d. The teacher who is regular and discriminating in her use of parallels so that minor subjects and reviews are made use of to best advantage to develop full concentration upon computations and consequent accuracy in them and to give ample training in the former gives evidence that she has a form of power that makes for skill—power to approach a solution of the problems of accuracy and practical acceleration.

PART II

COMMON FRACTIONS

CHAPTER I

INTRODUCTION

A. Objective Work in Fractions

1. The Essential Features

Objective work in number has for its aim to give the pupil the opportunity to learn number language in connection with its use in acts of measuring. It is to give him experiences in constructions that associate number terms with the measuring acts which they signify. It is to provide beyond question that when the language has found its place in the child's working vocabulary it will be vivid and full of meaning so that he will be able to use it intelligently and confidently.

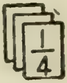
The test of the value of any number method, therefore, is the character of the mental products of its constructions. Are the concepts which they develop true to number? Are they complete and clear? Concepts contain no qualities that are not in every percept. If the latter are false or incomplete or not perfectly clear-cut, the concepts will have the same defects. When, for illustration, the pupil is required to construct 4, if all of the qualities that are essential to the concept 4 are not brought out in the construction, the objective work is defective and fails of its proper purpose.

When we speak of the language of number and complete and distinct imagery, we must regard the term as including fractional as well as integral number. We must have in mind number not alone in its use as expressing quantity but in its use as signifying processes. Arithmetical operations—operations in addition, subtraction, multiplication, division, and partition—to be used intelligently, must be understood. The pupil must know from experience in measuring operations how each affects quantity. There are percepts and concepts of number *processes* and these too must be developed by constructions—constructions which develop mental pictures that are true to the nature of the processes as well as to that of the numbers involved.

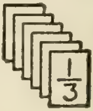
It is to this test as to the character of the concepts that grow out of it that a method—any method—of objective work, whether it relates to integral or to fractional number must measure up. It is method's standard of valuation, its yardstick. If a plan of number instruction meets this measure, it is safe. If it fails to meet it, it is defective.

2. *Its Relation to That in "Number Facts"*

In developing a fraction method one must determine whether there is anything in the fraction nature that requires special forms of concrete work. In other words, will the construction forms that the child used in the days when he was learning the "facts" of number, the number and the process forms, be of any use to him now?

The nature of a number is the same whether we apply it to integral or to fractional quantities. To illustrate: in the two numbers three (3) and three fourths (¾), three (3) means a group of three *integral* units—*a three;* three fourths (¾) means a group of three *fractional* units—also *a three.* The former expressed concretely is , the latter is concretely 

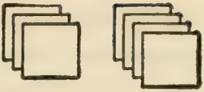
.

The two numbers differ only in the quantitative value of their measuring units. In the same manner, six (6) and six-thirds ($\frac{6}{3}$). The former is , the latter 

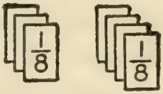
—a purely quantitative difference.

The significance of measuring operations also is preserved in fractions.

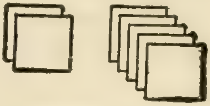
Three and four $(3+4)$ expressed concretely is . . . .



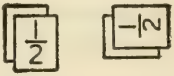
Three eights and *four* eights $(\frac{3}{8}+\frac{4}{8})$ is . . . .



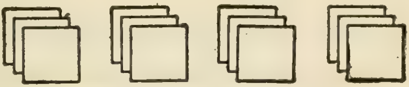
Seven less five $(7-5)$ is . . .



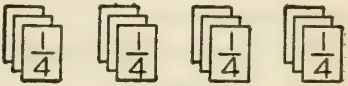
Four halves less *two* halves $(\frac{4}{2}-\frac{2}{2})$ is . . . .



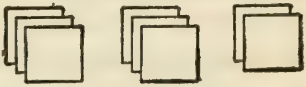
Four times three (4×3) is



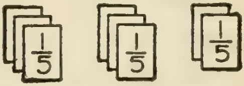
Four times *three* fourths $(4\times\frac{3}{4})$ is . .



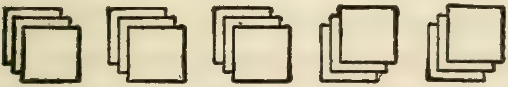
Eight divided by three $(8\div3)$ is . . . .



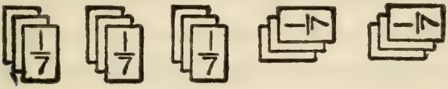
Eight fifths divided by *three* fifths $(\frac{8}{5}\div\frac{3}{5})$ . . . .



Three fifths of *fifteen* ($\frac{3}{5}$ of 15) is



Three fifths of *fifteen* sevenths ($\frac{3}{5}$ of $\frac{15}{7}$) is



All this goes to show that as we pass from integral to fractional and from fractional to integral expressions neither the nature nor the significance of the measuring operation changes. Since, therefore, number expressions integral and fractional are inherently identical in all respects excepting the character of the units, it follows that true number concepts can not be developed except as the objective forms and the language of the work in "number facts" is extended unchanged to fractions.

3. *The Aim of Fraction Method*

In the development of "number facts" there were three aims:

(1) The establishment of the group-unit nature of number.

(2) The spoken and written language of number operations (counting) in "and" (+), "times" (×), "take away" (−), "has how many" (÷), and partition ("of"), and the significance of " = " under these various measuring processes.

(3) The memory of the "facts" from long continued seat exercises in constructions.

These having been developed for integers are not development *aims* in fractions; because, as we have seen in 2, the nature of numbers, the significance of the operations, and the "number facts" are the same for fractional numbers as for integers. The aims in fraction work, therefore, are necessarily confined to—

(1) The nature of the fraction—its relation to the interger.

(2) The fraction terms.

(3) Fraction constructions and operations.

B. The Meaning of the Term Fraction

There are two classes of units—integral and fractional. Corresponding to these there are two classes of numbers, one made up of units that are wholes, the other made up of units that are smaller than a whole. Numbers made up of wholes we call integers. A number smaller than a whole we call a fraction. When we say whole number we mean a number made up of units that are wholes. When we say that a number is a fraction we indicate the kind of units that it contains, but nothing as to the quantity that it expresses. Quantitatively it may be very large or very small. When also we speak of a number smaller than a whole, we simply indicate the comparative value of its component units—units smaller than wholes.

It has been proven by experience that in this matter of objective fraction work many teachers before taking it up must have their attention called to this fact that the term fraction relates to the form of component units and not to considerations of comparative quantity; that ¾ and $\frac{40}{4}$ are fractions, not because of the quantities that they express, but from the fact that the measuring unit in each (¼) is a fraction. This is easily accounted for when we consider that in the ordinary computations of practical life improper fractions rarely occur. We are continually making use of such forms as $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, etc., but seldom of the improper fraction forms $\frac{3}{3}$, $\frac{8}{3}$, $\frac{11}{4}$, etc. The consequence is that many persons get into a state of mind in which their conception of "fraction" is that of a quantity smaller than a whole. As time goes on and no occasion arises with them for submitting it to the test of reason, association causes this impression gradually to deepen and to become sometimes tenacious, so much so that their special attention must be called strongly to the facts in order to convince them of their error.

A number is an integer or a fraction according to its make-up. In $6, for illustration, the measuring unit is $1, an integer. In $\$\frac{12}{2}$, the measuring unit is $\$\frac{1}{2}$, a fraction. These two numbers are equivalents. The one, however, is an integer and the other a fraction entirely because of the difference in their make-up—the form of their measuring units. $\frac{3}{4}$, $\frac{5}{9}$, and $\frac{300}{6}$ are fractions wholly because in each the measuring unit is a fraction. The *form* of this unit, therefore, and not the *quantity* expressed is the determining factor.

This is a simple matter upon which discussion would seem to be superfluous were it not for the fact that any form of construction work for the proper development of fraction imagery and language must bring it immediately into the foreground. More than this, the plan of introductory work in this subject that does not have for one of its definite aims the establishment of the "improper" as well as the "proper" form in the child's first "fraction" picture, will result in a narrow, imperfect conception of the term.

The teacher must, therefore, give serious consideration, in her plans for fraction development work, to the fact that while "fraction" etymologically signifies a *part* of a whole, a fragment, a number smaller than a whole, it has come to have a derived meaning under which it may refer to any number that is fractional in form, regardless of the comparative quantity that it may express, and that this inclusive meaning must stand out clearly as a construction feature. In outlining her number language plans, she should understand that this inclusiveness of meaning that has been given to "fraction" may be extended with perfect propriety to its simple parallel term "number smaller than a whole," which is so necessary as the introductory term in the development, and that this term may also refer to *form*—may be made to include, like fraction, any number whose measuring unit is smaller than a whole. In this use of the expression, $\frac{3}{2}$ or $\frac{8}{4}$ or $\frac{24}{6}$ would be a number smaller than a whole in the same sense that it is a fraction.

It is to establish this interpretation of fraction that the teacher at the very beginning of fraction construction asks for "a number smaller than a whole" and then for a *different* number smaller than a whole. The responses to these questions as shown in the development outline (see Chapter III, *A*) indicate that the pupil will easily make the inference which these successive questions invite—that the expression is to be taken broadly, the reference being entirely to the *form* of its measuring unit. Thus is established at once the proper interpretation of this preliminary expression, number smaller than a whole—an interpretation that the term fraction will, when introduced, inherit.

C. Objects

1. Must be Prepared by the Pupil

The development of fraction imagery begins with the construction of the fractional units from wholes. The child that does not have practical experience in preparing in this way his own objects loses a very important part of fraction training. The meaning of half, third, or fourth, is not clear and full except to the child that has constructed the objects and proved that they reasonably meet the tests of accuracy—tests that he himself understands and can apply. The work of construction concentrates the thought upon the conditions which determine the fractional nature and name of the object and makes these conditions a part of the imagery.

The rule in the preparation of the objects should be that each child prepare his own supply. An exception, however, to this rule is the final *working* objects of a child who may be so clumsy with his hands that those of his own making do not satisfy the tests of equality. The constant use of such objects would make for faulty imagery and habits of careless thinking. Such children should have their real working objects made for them by other pupils or by the teacher. They should, however, have much experience in *trying* to make fractions of all kinds. This experience, because they are *trying* to make parts that are equal to each other and because of the concentration of mind at the same time upon this necessary equality, will serve all practical purposes in the development. The fractions that they make, the imperfect ones, will find their way into the waste-basket to be replaced for their regular fraction construction work by objects that are reasonably perfect.

2. *Material for Objects*

When we accept the principle that the child must make the fractions to be used in his construction work, we practically limit ourselves to object material that may be folded and torn into equal parts. To construct halves, thirds, fourths, fifths, sixths, sevenths, etc. from 10-inch or 6-inch or other splints or kindergarten sticks or from pasteboard or other stiff material that will not fold or crease without breaking is manifestly beyond most children of the fraction age. To say that he might mark off on such objects with the aid of the foot-rule the lines along which to cut them for making the fractional parts would assume that the pupil has had fraction experiences which this object work is designed to develop. Such use of a foot-rule would require a fair knowledge of fractions. Again, the difficulty of marking the objects so that one could know quickly which pieces are from the 10-inch and which from the 6-inch or other stick, and what the fraction value of each object without tedious and unprofitable delays for comparison and measurements would eliminate from consideration, except for limited supplemental use, the splint, the kindergarten stick, and other similar objects even if the problem of measuring off the fractional parts were solved.

It is very desirable, however, that each child have a small supply of a few fractional units—halves, thirds, fourths, sixths, and eighths, possibly, from one kind of splint and one kind of other object (a kindergarten stick, perhaps) to break the monotony of paper objects. In one of these classes of objects the child might have halves, thirds, and sixths; in the other, halves, fourths and eighths. Their use would, of course, be limited but their occasional use would put variety into the percepts and thereby contribute to the development of better concepts. This is what is meant by "limited supplemental use" to which reference was made above.

3. *Variety of Objects*

General fraction imagery can not be developed through objective work with one class of units. With a single unit—oblong paper wholes, paper circles, square paper wholes, 10-inch splints, or 6-inch kindergarten sticks—the child's imagery of halves or fourths or thirds or other fractions would be a paper or a splint or a stick of a certain size and shape. He would not have the general notion half, third, or fourth, the fraction development work should secure.

On the other hand, if we use a variety of wholes—oblong papers of different sizes as 6″×7½″ and 4″×6″; paper circles; 10-inch (or 12-inch) splints; and 6-inch or other kindergarten sticks—with several halves, several thirds, several fourths, etc. from each of these classes of objects, there will be no opportunity for the child's notion of half or third or fourth to grow out of objects of any given size or shape or material. He will have in this varied assortment of halves, thirds, fourths, etc., proper material for the development of fraction concepts.

4. *Quantitative Equality of Objects*

The objects used in the construction of any given integral expression—3×4, $2 + 5$, $6 - 4 =$ $9 \div 5 =$, or $\frac{2}{3}$ of $12 =$—must be quantitatively equal. (See Vol. I, pp. 33–37.) It should not be necessary to say that the same principle applies to fractional expressions. In $\frac{3}{4}+\frac{5}{4}$, the fractional units must be from equal wholes. In $4\times\frac{6}{3}$ the measuring units are thirds of equal wholes.

The child, as has been shown, should be supplied with a variety of objects but the individuals of each of these classes of objects must be equal, else the fourths or sixths or halves made from them would be unequal and not proper construction material for the development of accurate mathematical imagery. If two or more kinds of paper circles are used, the individuals of each kind must be equal in diameter and thickness.

5. *Marking the Objects*

With an assortment of rectangular paper wholes which differ but little in size (5″×7″, 6″×7½″, 4″×6″), the child would have his objects in confusion and be unable to distinguish the pieces as to fraction values without measuring. This would lead to errors and loss of time in construction work with no compensating advantage.

To avoid this, pupils should mark all wholes of one kind A, all those of another kind B, all those of another kind C, etc. Thereafter papers would be distinguished by their letters —A papers, B papers, C papers, etc. When fractions are made from A papers, the child should at once mark each piece with its proper letter and fraction name—as A $\frac{1}{8}$, A $\frac{1}{3}$, A $\frac{1}{16}$, etc. He should in the same manner mark fractions from his B, C, and other papers that he may have. If two kinds of circles or splints are used they should be marked in the same way. All fractions should be marked with their fraction names. If only one kind of splint or paper circle is in use, each fractional unit should be marked with its proper mark—$\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{5}$, etc. This will contribute to more rapid construction work.

6. Economy in Objects.

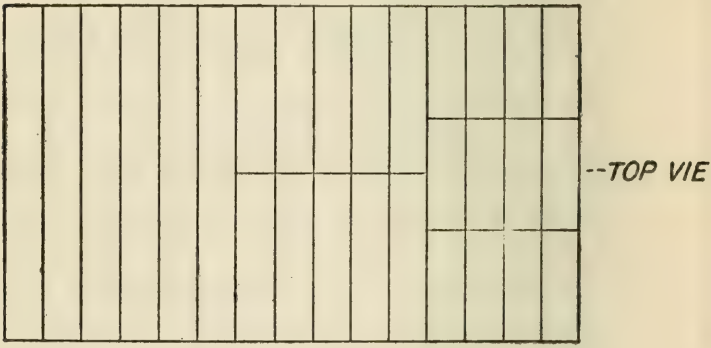
It has been shown that the essential feature of the various fractions from each kind of whole is equality. If halves or fourths or other fractions from a given whole are not equal they are not proper for use.

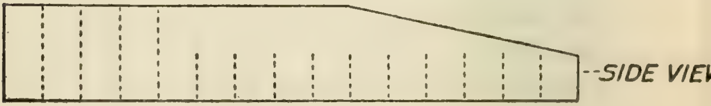
It is not difficult or expensive to secure rectangular paper objects. This is the ordinary shape of school papers of all kinds. Circles, however, are not obtainable without liberal payment for the "circular" and the "equality" features. Circular objects are not necessary. There may be a sufficient variety of objects without this kind.

7. *Receptacles for Pupils' Objects*

Objective work in fractions involves the use of objects that are not as easy to take care of as pegs and inch-sticks of the primary grades because of their size—length and width. Added to this is the fact that there are several denominations of objects (halves, thirds, fourths, fifths, etc.) and several kinds of each of these denominations; for example, A, B, and C halves; A, B, and C thirds; A, B, and C fourths.

The problem before the teacher, therefore, is a means of caring for the objects—of keeping the denominations separate. No best method can be suggested. Some teachers bind each denomination (the A, the B, and the C halves; or the A, the B, and the C thirds, etc.) in one bundle held together with a small rubber band, —thus putting all the halves together, all the thirds together, all the fourths together, etc. These bundles while in use lie on one end of the pupil's desk; when not in use they are put into large envelopes furnished for the purpose. Others have pasteboard boxes made 7″×12″, 2″ deep. These are divided by pasteboard partitions into 15 sections, each section approximately ¾″×7″. Some of these sections are left full size; others are separated into two sections, each ¾″×3½″. Others are made into three sections. Each denomination of fraction objects is assigned its special section—halves in the first, thirds in the next section, etc. If no fractions are used of the denominations 11ths, 13ths, 17ths, 19ths, 23rds, 26ths, 29ths, 31sts, 34ths, 37ths, 38ths, and 41sts, the sections provide places for all other denominations to 42nds. These are all the fractions required for all practical development work. The following is a diagram of such a box:





Another device is a 10″×12″ cardboard with cloth pockets like apron pockets, the several bundles of fractions bound with rubber bands being kept each in its assigned pocket.

8. Mixing Fractions of the Same Fractional Value

The child has made and marked a dozen or more fractions of each denomination—halves, thirds, fourths, fifths, etc., from each kind of wholes. He has all fractions up to 16ths or 20ths (excepting 11ths, 13ths, 17ths, and 19ths) from his A, his B, and his C papers, and from his other wholes if he has any. All these halves should be put together into the "half" section or pocket of the object receptacle—all the thirds into the "thirds" section or pocket, all the fourths into the "fourths" section or pocket, etc.

This mixing of the fractions of each denomination is necessary in order to train the child to recognize that he must select for each number or expression measuring units from the same kind of whole. To construct $\frac{5}{8}$, he must choose all A 8ths or all B 8ths or all C 8ths or all 8ths from some other class of wholes—splints or kindergarten sticks, for instance. The same rule must apply to an expression like $\frac{3}{5}+\frac{7}{5}$—the expression must involve 5ths of one class of integral units. In making $\frac{2}{3}+\frac{4}{2}$ he will choose the thirds and the halves from his A fractions or from his B fractions or from his C fractions or from some other class in the same way. It is the objective method of teaching the child that in any arithmetical expression in +, −, ×, ÷, or "of," there should be but one kind of measuring unit.

In the oral work in connection with it the teacher asks, "How did you make the expression?" ("I made it with A papers" or "I made it with splints" or "I made it with C papers.") "What is the rule?" ("The rule is that I must make an expression with one kind of objects—all A papers or all B papers or all C papers or all some other kind of objects.")

9. Fractional Units that are not Necessary

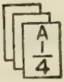
The aims of fraction development work are to give the child experience in fraction construction, in the expression of fractions, in the changing of fractions from one form or name to another, and in fraction measuring processes. To know how to construct fraction expressions—$\frac{2}{3}$, $\frac{5}{13}$, $\frac{7}{9}$, $\frac{8}{17}$, $\frac{4}{5}$, $\frac{7}{12}$, $\frac{4}{11}$, etc.—it is not necessary that the pupil shall have actually made every one of these kinds of fractional units. If he has repeatedly made and used halves, thirds, fourths, fifths, sixths, sevenths, eighths, ninths, tenths, and twelfths, he will know just as clearly what is meant by 17ths, 13ths, 25ths, 29ths, and 23rds as if he had made and used them. The same is true of the statements and processes. If the child knows from experiences with the fractional units to 12ths what is and what is not a mixed number, what the nature of the measuring processes of +, −, ×, ÷, and "of," he will know as clearly what other mixed numbers are and the measuring processes which these signs indicate when applied to fractional units smaller than 12ths, as if he had had actual concrete work with every fractional unit to 100ths. Objective work is for the purpose of learning principles; and these, mastered with reference to fractional units to 12ths, are mastered with reference to all fractions.

If this is true, the teacher may safely confine her fraction construction and her measuring work to expressions which offer the least difficulties. The fractional units to 12ths are not difficult to make and all of them except perhaps 11ths should be used. Of the fractional units smaller than 12ths, the following need not be used in the development work: 13ths, 17ths, 19ths, 22nds, 23rds, 26ths, 31sts, 33rds, 37ths, 39ths, 41sts, etc. Fraction imagery will not be incomplete through such omission. Of the fractional units smaller than 40ths, few, if any, need be used in the concrete work. There may be occasional use *in constructions* for 42nds, 45ths, and 48ths, but the dictations should be given so as to avoid them as much as possible and to avoid entirely the use of smaller *objects* than 48ths. Later, when dictations are given for work without objects, it will not be necessary to consider construction difficulties, and fractional units smaller than 48ths will be brought into occasional use.

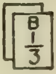
10. Grouping Fractional Units

The examination of the concrete work of a class of children may be very much simplified by requiring a proper arrangement of the objects in the groups.

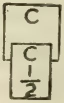
$\frac{3}{4}$ should be expressed



$\frac{2}{3}$ arranged in proper concrete form is



$1\frac{1}{2}$ in proper concrete form is



This plan of grouping so that the several units of the group project a half inch or an inch at the top enables the teacher to choose one or two positions in a room from which she may see the number of objects in such group on every desk. The work of inspecting a construction may thus be done in a few seconds.

11. "Making" and "Exchanging" in Fraction Constructions

It will readily be seen that if the child should make new fractional units for each exercise in the series work under Chapters XIII–XV there would be a very large consumption of paper. There would be an accumulation of halves, thirds, fourths, etc. beyond any possible need for his later work.

Now, there is no advantage, from any standpoint, in this actual construction of new units each time. If a child, instead of tearing the half, exchanges it for two fourths, he loses nothing in imagery. There is no special pedagogical value in the fourths that are made that is not in the fourths taken from the supply box to replace the half. In the one case the half makes two fourths, in the other it is equivalent to two fourths. Either view satisfies the aim of the object work.

In the interest, therefore, of economy of material, the "making" should not continue beyond the time when the child has enough fractional units of each kind for his ordinary construction needs. After that the child "exchanges" for his constructions and adapts his oral language to acts of exchanging.

"I take a half and exchange it for two fourths, I take another half and exchange it for three sixths, etc."

CHAPTER II

PRELIMINARY VOCABULARY

Topics:—Number and numbers. Unit and units. Whole.
Whole numbers.

THE development of fractions properly begins with the following number terms: number and numbers, unit and units, whole, whole numbers.

A. Number, Numbers

AIM.—The child has used these terms in his number work from the very beginning in a general way. Before using them in the fraction development steps here outlined, they must be renewed and made definite. This first step, therefore, is to give them the vitality that they must have to lead the child to apply them unhesitatingly when the development brings him for the first time to constructions with units which he afterwards learns to call fractions.

METHODS OF PRESENTATION

(1) Objects in the child's hands.

The work follows in general the simple group-unit plan outlined (Vol. I, pp. 56–62) for Grade I except that there the aim was the development of the group-unit* nature of number, here the aim is the development of the terms number and numbers, the group-unit being an incidental aim.

"Show me a number." (Most pupils will hold up 1.)

"What is your number?" "Yours?" "Yours?" etc.

"Show me another number." (Other numbers than one will now appear.)

"What is your number?" "Yours?" "Yours?" etc. (The pupils called upon in this case should be those who are "showing" a number other than one—2, 3, or 4. The purpose of this is to lead the children away from a notion that some of them may have, that 1 is the only "number." The quiet recognition of 4, 2, 3, etc., as numbers, establishes at once that 1, 8, 4, 6, 2, etc. are all "numbers" in the meaning of the term.)

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* The group-unit nature of 2, 3, 4, 5, etc. is of utmost importance and must be insisted upon absolutely in all constructions. (See Vol. I, p. 46 (*a*), (*b*), (*c*).) If a child in response to "Show me 2" holds up two objects *with a finger between them*, he is showing 1 and 1, not 2. If a child holds up three units in contact with each other, it is a group, 3. If a finger intervenes it is not a single group. It is two groups 2 and 1, or 1 and 2. Accurate number imagery requires insistence upon groupings that are true to number—the constituent units of a group being in actual complete contact. Neglect or carelessness in this will defeat development steps later on.

"Show me a different number" (different from what each has shown)?

"What is your number?" "Yours?" "Yours?" etc.

"Show me *two* numbers."

"What are your numbers?"* "Yours?" "Yours?"

"Show me a number as large as both of the numbers you have just shown me."

"What is your number?" "Yours?" "Yours?"

(The above suggestions indicate the form of the class work. It will be necessary to give but a very few minutes to this step.)

(2) Objects in the teacher's hands.

This is the reverse of (1)—the teacher mak-

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*It is important that the teacher should require a pupil to read such constructions from his left to his right. If a child holding up 2 in his left and 3 in his right hand should speak of them as 3 and 2 his reading is wrong. He needs instruction in the direction in which concrete number is read. When the teacher facing the class holds up two groups— two numbers—she must see that the child reads them from his left to his right. The *teacher* must read them from the *child's* left to right in the same way.

ing the constructions before the class, the pupils applying the terms number and numbers. The pupil's answer to a construction should be, "You are showing me a number"; or, "You are showing me two numbers"; or, "You are showing me three numbers"; etc. The teacher should ask after every such reply ("You are showing me a number" or "You are showing me two numbers"), "What is the number?" or "What are the numbers?"*

After a few constructions the pupils should respond in full, "You are showing me —— numbers. The numbers are —— and ——." This full response should be given without the question by the teacher, "What are the numbers?"

B. Unit, Units

Aim.—To review these words (already in the child's vocabulary) in order to make them available, when, in the development plan later, we tempt the child to apply them to fractional objects. The words are familiar to the child through previous number experiences but must be freshened up for ready use in the emergency of a development step later on.

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*The teacher must be consistent in requiring pupils to read her constructions from their left to their right.

METHODS OF PRESENTATION

(1) Objects in the pupils' hands.

"Show me a unit." (The response to this must be *one object*, not a *group* unit.) No explanation will be necessary. Without doubt each child will respond with *one*. It can hardly be conceived that any child will respond with a *group*.

"Show me a number having three units." (The child's response to this must be three units *in a group* as in "Number" and "Numbers" above.)

"What is your number?" "Yours?" "Yours?"

"Show me two numbers,* one having two units, the other three units."

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* The teacher must note the form of the questions. She does not ask for two *units* or three *units*, but *for numbers having two units* or *three units*, etc. If the dictation were in form "Show me two units," the child's response would properly be 1 and 1, not a group 2. She will note also that immediately upon the introduction of "unit," the terms "number" and "numbers" are brought into use in connection with it. Unit and number are now to be kept in constant use. She will bear in mind also that responses—the constructions—to such dictations should be accurate, the groups held up or arranged on the desk so that when read from the pupil's left to his right they will conform to the dictations. A construction arranged otherwise must be

"What are your numbers?" ("Two and three.") "Yours?" "Yours?"

The above suggestions indicate the form of the presentation. Five or six additional dictations are all that will be necessary for its completion. This section requires in all not more than three minutes of time.

(2) Objects in the teacher's hands.

The teacher constructs, the pupils state of each construction whether it has one, two, or three numbers, and how many units each of the numbers has:

("You are showing me two numbers, one having —— units, the other —— units; or, "This is two numbers, one having —— units, the other —— units"; or, "This is three numbers, one having —— units, another —— units and the other —— units.")

C. Whole

AIM.—To introduce the word briefly before we take up *whole number*. This work in connection with "number" and "unit" must be brought into use *with objects very briefly*, preparatory to the introduction of *whole* number.

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declared to be *wrong*. A dictation, "Show me two numbers, one having four units, the other five units," must not be accepted when the response is a *five* and a *four* when read from left to right.

It would not be proper to throw *whole number* into use without first giving one moment to *whole.*

METHOD OF PRESENTATION

"Show me a whole." (The pupil's response to this must be one object not a group.)

"Show me a number having two wholes." "What is your number?" "Yours?"

"Show me two numbers one having three wholes the other two wholes."

NOTE.—Not more than one or two minutes need be given to this step. The teacher after two or three dictations as above passes to whole number.

The teacher passes at once without pause from the last dictation here to the first dictation under *whole number* below. The more abrupt the passage to that dictation the clearer and better the development.

D. Whole Number

METHOD OF PRESENTATION

(1) Objects in the pupils' hands.

There should be no formal introduction of the term. It should be *thrown* into use—dictated *abruptly* as if it were a term with which all are familiar. Following closely upon the last dictation under "whole" above—

"Show me two whole numbers each having three units." (The pupils will show 2 3's.)

"What are your numbers?" "Yours?" "Yours?"

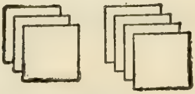
"Show me two whole numbers, one having four units, the other two units."

"What are your numbers?" "Yours?"

Other dictations of the same kind follow. There are no explanations. The child readily infers that 3 is *a* whole number, that 4 is *a* whole number, thus continuing the group-unit imagery of his previous training.

(2) Objects in the teacher's hands.

The teacher makes the constructions, the pupils stating in response what each construction is*—two whole numbers, three whole numbers, or one whole number as the case may be—and how many units in each.

The teacher holds up 

("This is two whole numbers, one having three units, the other four units;" or "You are showing me two whole numbers, one having three units the other four units.")

We have now introduced all the terms that are needed in connection with the development of the fraction idea. The terms number, unit, whole, and whole number have been renewed in the vocabulary. With this equipment of number terms and imagery, "numbers smaller than wholes" then "fraction" are next in the sequence of development steps.

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*The teacher, as has been suggested, must be insistent upon proper left to right reading in these exercises.

SUMMARY.

1. Exercises to revive the terms *number* and numbers. (*A*.)
   The group-unit nature of numbers must not be overlooked in constructions. (*Note 1*.)
2. Exercises to bring unit and units into use in objective work. (*B*.)
3. Exercises to introduce whole and whole numbers. (*C* and *D*.)
4. The need of constant use of the terms. (*D*, *Note 2*.)

Time required for the work of this chapter three days.

NOTE.—The teacher must bear in mind that while the terms "number," "unit," etc. are fresh in the vocabulary they will not necessarily remain there in an active state unless they are frequently brought into use by exercises like those which have just been given—"Show me a number having —— units"; or, "Show me two whole numbers, one having —— units, the other —— units." There is a fundamental rule in education applicable here that no teacher should lose sight of: that, in the teaching of any subject, nothing that has been development should be permitted to fall into disuse even for a short time.

CHAPTER III

THE TERM FRACTION

Topic:—The construction conditions for a number smaller than a whole—a fraction. Fractional number.

A. Number (and Numbers) Smaller than a Whole

AIM.—(i) To develop that the parts of a whole, in order that they may come within the meaning of the term "number smaller than a whole," must be equal to each other. The center of attention is to be held upon the *equality* feature.

(ii) To count "numbers smaller than a whole"—one, two, three, etc.—and to apply to them the terms "unit" and "number" as in the exercises in Chapter II.

METHOD OF PRESENTATION.

(1) Objects in the hands of the pupils.

There is no formal introduction. The step follows work like that in "Whole Number" directly, abruptly—without explanation or other intervening work.

After a few exercises in Chapter II, *D* (2) the teacher goes back to *D* (1) and gives two or three dictations actively and with vigor in order that the pupils' minds may be brought into the line of thought that we wish them to follow through the development. The following dictations indicate the form and sequence of these dictations. *The teacher will observe* that the dictation below immediately before that for "number smaller than a whole" is one that involves a *one*, 1 whole.

"Show me a number having 3 units." "Papers down."

"Show me a number having 4 wholes." "Papers down."

"Show me a number having 2 wholes." "Papers down."

"Show me a number having 1 whole." "Papers down."

"Show me a number smaller than a whole."

To this all the pupils will respond with parts of a whole. Practically all will tear the paper into two parts. Some will tear into *equal* pieces, others into two parts carelessly. Those who tear into *equal* pieces will do so not because they know or think that the parts must be equal but because of habits acquired in school or at home of doing things neatly. Nearly all will "show" *one* of the parts. Whether the pupil shows one or both of the pieces is immaterial.

This is the teacher's opportunity to drive home the lesson of *equality* of the parts.

This is driven home best by *indirect* teaching. To impress it indirectly, the teacher notes two or three pupils whose parts are *unequal* and two or three who seem to have made their parts *equal.* She goes first to a pupil whose parts are unequal and asks, "What is your number?" The pupil will answer "one" or "two" according to what he is "showing." The teacher says, "I will see whether yours is a number smaller than a whole." She takes the piece or pieces of paper held up and with it or them the other piece of the whole, if any, and in full view of the class begins carefully to put the pieces together corner to corner and edge to edge. She holds them so that the inequality may be seen, points to the inequality, and merely says, "No, these are not numbers." Then she goes to another child that has unequal parts. "What is your number?" (The child states the number.) The teacher then takes the piece or pieces "shown" also the other part or parts of the whole, if any, puts them together, corner to corner and edge to edge as before, and repeats, "No, these are not alike, these are not numbers smaller than a whole." Then she goes to a child whose pieces she feels sure are equal and asks, "What is your number?" as before. She then takes the pieces in her hands and compares them by placing them corner to corner and edge to edge as before. Then she holds them so that everyone may see that they agree in size and says, "Yes, these are right." "You showed me a number smaller than a whole." Then she goes to another who has equal pieces and asks, "What is your number?" and takes the pieces as before and compares them placed corner to corner and edge to edge. "Yes, yours are all alike. You showed me a number smaller than a whole."

NOTE 1.—By this time the lesson will be thoroughly forced home and pupils all over the room will be seen comparing their own pieces, placing them corner to corner and edge to edge to see if they are equal, and approving or condemning their own constructions according as they find that their pieces are equal or unequal. Those who find that their constructions are unequal will be seen either laying aside their pieces and making new ones or trimming up the old ones by tearing off a part of the larger piece or pieces to make them equal.

The teacher goes from one to another with questions and demands for proofs—"What is your number?" (The child replies.) "Are you sure that yours is a number smaller than a whole?" ("Yes, my pieces are equal.") "Prove it to me." (The child puts his pieces very carefully together corner to corner and edge to edge to make this proof.) "Yes, you showed me a number smaller than a whole."

NOTE 2.—The teacher must be somewhat exacting about the equality of the pieces which are made from the wholes.

(*a*) Assisting the Clumsy Child.

If in tearing the *whole* the tearing line should be jagged, the parts are not equal and the paper should be laid aside and not used in constructions. The child must take a new paper and construct again. Some pupils (a few who are clumsy with their hands) will not be able to make equal pieces. These children should *try* to make them, even at the loss of several pieces of paper during a lesson. *Trying* to make equal pieces establishes the equality imagery. As to the objects that he keeps for regular use, he must have help in making them so that he will have pieces that are not unequal. The pieces that are accepted for class use as "numbers smaller than a whole" should be as nearly perfect, as to *equality* of the parts of any single whole, as careful folding and tearing can make them. All this for the sake of emphasis on the *equality of the parts*.

(*b*) Trimming off the larger to make the parts equal.

There is a feature of the construction work to which the teacher must give attention. The development has brought out that the parts of a given whole are not "numbers" unless they are *equal*. Now, there are two ways of securing equality, as the teacher will discover at the outset — (i) by tearing into equal pieces and (ii) by trimming down the larger of the unequal pieces. Children will be seen attempting to secure equality in that way. The teacher should go to some pupil whom she has seen making equal pieces by trimming—

"What is your number?" (The pupil replies.)

"Are you sure that yours is a number smaller than a whole?" ("Yes, my pieces are equal.")

"Prove it." (The child compares the pieces and shows them to be equal.)

The teacher now takes the pieces and applies them to a whole of the same kind and finds that the *pieces* do not cover the *whole*, they are too small.

"Is all the paper here?"

"Did you trim off any of it?"

"Did you throw away some of the paper?"

These questions will bring out an admission that the pieces were trimmed up to make them alike.

"You mustn't tear off a bit of the paper."

"The pieces must have *all* the paper in them."

"You would better put the pieces in the waste-basket and try again."

"You must not trim the papers."

Another child who has trimmed up his pieces must be approached in the same way until it is fully impressed upon the class that papers that are not right without trimming must be laid aside and the construction made with a fresh paper. The teacher must be sharply on the lookout for papers that have been "adapted" in this way. It is a common practice at first to make parts equal in this way. It is the opportunity to teach objectively not only that the parts of a whole must be *equal* but that they together are equivalent to the whole.

"Show me another number smaller than a whole." (Some who before "showed" one will show *two* now, others who showed *two* will now show *one*. Others may tear another paper and show perhaps three or four pieces.)

"What is your number?" "Yours?" "Yours?"

"Are you sure that yours is a number smaller than a whole?"

"Prove it to me."

"Is your paper all there?" "Have you trimmed off any of your pieces?"

"Show me a number smaller than a whole having 3 units."

"What is your number?" (The child will answer, "Three.")

"Prove that yours is a number smaller than a whole."

"Show me two numbers smaller than a whole,—one having 2 units, the other 3 units."

"What are your numbers?" "Yours?" "Yours?"

"Show me two numbers smaller than a whole,—one having 4 units, the other one unit."

"What are your numbers?" "Yours?" "Yours?"

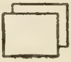
"Show me two whole numbers,—one having 2 units, the other 4 units."

"What are your numbers?"

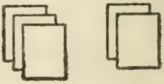
NOTE 3.—Other dictations should be given, now in whole numbers, and again in numbers smaller than a whole. If in responding to dictations for two or more numbers smaller than a whole the child should "show" in one of the groups pieces of a certain size and in the other (or one of the others, if there are three) pieces of a different size, the construction must be accepted. If, for illustration, one of the groups in a group construction should have *three* of the *four equal parts* of a whole and the other *two* of the *six equal parts* of a whole, the child would properly answer in reply to the question as to what his numbers are, "Three and two." This is right. We asked for a count of the *units in each group* and that count was *correct.* We are not using the names of the fractions now—thirds, sixths, fourths, etc. These names come later. The only construction that can be objected to is a group made up of units of two or more sizes. If such construction should be discovered the teacher should change the construction for the child *without comment or suggestion.* An explanation, suggestion, or reproof might embarrass the teacher in the development of mixed numbers later.

(2) Objects in the hands of the teacher.

The teacher constructs, the pupils state orally what the constructions are.

The teacher holds up 

("You are showing me a number smaller than a whole having two units.")

The teacher constructs 

("You are showing me two numbers smaller than a whole, one having three units the other two units.")

This shows the form of the exercise. It is merely the reverse of (1).

B. Fraction

Aim.—To introduce "fraction" as the proper word to apply to a "number smaller than a whole."

METHOD OF PRESENTATION

(1) Objects in the hands of the pupils.

"I am going to give you a new word. It means a number smaller than a whole."

"The word is fraction."

The teacher writes *fraction* on the blackboard.

"Hereafter we are going to say *fraction* when we wish to speak of a number smaller than a whole."

"Show me a fraction having three units."

"Show me two fractions—one having two units, the other four units."

Note 1.—The dictations for constructions in the form as suggested in A(1) should be given here excepting that the term "fraction" should be used in the exercises instead of "number smaller than a whole."

The teacher, however, *must not* ask the question, "What is your *number?*" or, in the form that it would naturally and almost necessarily take in this section, "What is your *fraction?*" When we ask for "fraction"—"What is your *fraction?*"—the child's response must include not alone the *number* (the numerator) but the *name* of the fraction, the denominator. We have not yet developed fraction *names*—denominators—and of course must not (cannot) use the term. Hence this suggestion that the question be avoided. The work here is confined to the "show me" form.

"Show me two whole numbers,—one having one unit, the other four units."

"Show me two numbers,—one a fraction having four units, the other a whole number having two units."

"Show me three fractions,—one having four units, another three units, the other two units.

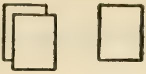
"Show me two numbers,—one a whole number with three units, the other a fraction with five units.

NOTE 2.—A very few such dictations are sufficient. Only a few minutes need be given it.

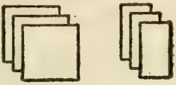
(2) Objects in the hands of the teacher.

The teacher constructs, the pupils state what the construction is.

("You are showing me a fraction having four units.")



("You are showing me two fractions,—one with two units, the other with one unit.")



("You are showing me two numbers, one a whole number having three units, the other a fraction having three units.")

Note 3.—The teacher must be insistent in all this work upon the pupil reading concrete expressions from his left to his right.

C. Fractional Number

Aim.—To lead the pupil to use this term interchangeably with "fraction."

METHOD OF PRESENTATION

(1) Objects in the hands of the pupils.

No explanation is necessary. The plan of presentation is to give two or three dictations like those in *B* (1) above as a preliminary, and then to put it into use at once abruptly—

"Show me a fraction having two units."

"Show me two fractions,—one having four units the other one unit."

"Show me a fractional number having five units."

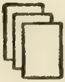
"Show me two numbers,—one a whole number with three units, the other a fractional number with three units."

"Show me a fractional number with four units and a whole number with one unit."

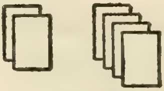
This shows the manner of introduction. It will not be necessary to give more than eight or ten dictations of this kind before passing on to (2).

(2) Objects in the hands of the teacher.

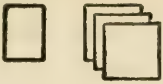
The teacher constructs for the pupils to state what the constructions are.



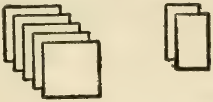
("You are showing me a fractional number having three units.")



("You are showing me two fractional numbers,—one having two units, the other four units.")



("You are showing me two numbers,—one a fractional number with one unit, the other a whole number with three units.")



("You are showing me a whole number with five units and a fractional number with two units.")

Only a very few such dictations are necessary.

SUMMARY.

1. Exercises in construction of "a number smaller than a whole," the proper introductory descriptive term for fraction.
    - (*a*) Constructions to bring out that such parts of a whole must be equal to each other (*A*, (1).)
    - (*b*) Proper assistance in preparing the objects. (*A*, Note 2 (*a*).)

(*c*) Trimming "parts" to secure equality. (*A*, Note 2 (*b*).)

(*d*) What may constitute *two numbers* in this development step. (*A*, Note 3.)

2. The term fraction introduced—without fraction names. (*B*.)
3. The term fractional number introduced as fraction synonym. (*C*.)

Time required for the work of this chapter, three days. This includes the time for review of Chapter II.

CHAPTER IV

THE TERMS INTEGER, INTEGRAL NUMBER AND MIXED NUMBER

Topic:—The introduction and definition of these terms.

A. Integer

a. AIM.—To put the term into the child's vocabulary. (It is almost necessary that this word come into daily use. The child is using "fraction" and "fractional" and it will facilitate expression very much if he is led to use their opposites—integer and integral.)

b. PRESENTATION.

(1) Objects in the pupils' hands.

"I am going to give you a new word, and I want you to learn to use it. The word is *integer*." (The teacher writes "integer" on the board.)

"Integer means whole number. Instead of saying whole number I want you to say *integer*."

"Show me an integer." (The pupil will doubtless show one object.)

"Yes, you are showing me an integer."

"Show me a different integer." (This is to invite the inference that 2, 4, 7, 3, etc., are integers.)

The teacher should make no explanation or suggestion other than to go to several who have "shown" an integer greater than *one* and approve what they have "shown."

"Yes, you are showing me an integer."

"Yes, you are showing me an integer."

"Yes, you are showing me an integer, etc."

"Show me two integers—one having four units, the other two units."

"Yes, you are showing me two integers—one having four units, etc."

"Show me three fractional numbers."

"Yes, you are showing me three fractional numbers."

'Show me two numbers—one an integer, the other a fraction."

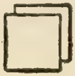
"Yes, you are showing me, etc."

"What is an integer?" ("An integer is a whole number.")

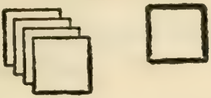
(Several dictations like the above should be given and the term "fraction," "fractional number," and "integer" used.)

(2) Objects in the hands of the teacher.

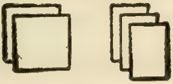
(The teacher holds each construction before the class.)



("You are showing me an integer with two units.")



("You are showing me two integers—one having four units, the other one unit.")



("You are showing me two numbers—one an integer having two units, the other a fraction having three units.")

It will not be necessary to give more than three or four dictations of this kind before passing to *B*.

B. Integral Number

a. Aim.—The introduction of the term.

b. Presentation.—The term does not require development or explanation. Its introduction should be abrupt after the three or four dictations in *A*, *b* (1).

(1) Objects in the pupils' hands.

"You may take your objects."

"Show me an integral number with three units."

(The inference that integral number is another expression for integer is so natural that the pupil will respond without hesitation. The transition from integer to integral number is as simple as that from fraction to fractional number.)

"Yes, you are showing me an integral number with three units."

"Show me an integral number having five units."

"Yes, you are showing me, etc."

"Show me two numbers—one a fractional number having three units, the other an integral number having two units."

"Yes, you are showing me, etc."

"Show me two numbers—one a whole number, the other a fraction."

"How many units in your whole number?"

"How many units in your fraction?"

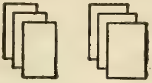
"Show me two numbers—one fractional, the other integral."

"How many units in the fractional number?" How many in the integral number?"

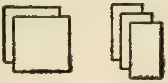
(2) Objects in the hands of the teacher.

The teacher constructs the expressions and holds the constructions before the class.

("You are showing me an integral number having four units.")



("You are showing me two fractional numbers each having three units.")



("You are showing me two numbers—one an integral number with two units, the other a fractional number with three units.")

(The above shows the form of the work. A few dictations, eight or ten, will suffice before passing to *C*.)

C. Mixed Numbers

a. AIM.—To develop the *objective* mixed number. The child is to learn to recognize a concrete expression as integral, fractional, or mixed. The *written* mixed number is, of course, a much later development.

b. PRESENTATION.—(1) Objects in the pupils' hands. The child understands what is meant by *integral number* and *fractional number* in the concrete. *Mixed number* is the very next element in the sequence and consequently we may depend upon him to infer what a mixed number is and to construct it, provided the teacher brings him squarely face to face with it.

The introduction is prefaced by three or four dictations under Chapter III, *C b* (1) and Chapter IV, *B b* (1) following which "mixed number" is an easy inference—

"Show me an integral number."

"What is your number?" "Yours?" "Yours?"

"Show me a fractional number."

"What is your number?" "Yours?" "Yours?"

"Show me two integral numbers."

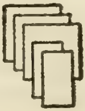
"Show me two fractional numbers."

"Show me an integral number."

"Show me a fractional number."

"Show me a mixed number."

The pupils will easily infer under this form of approach what is meant by a "mixed" number and will construct a group partly integral and partly fractional.

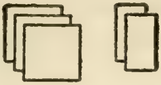


(Of course there will be almost as many mixed numbers in the response to this dictation as there are different pupils in the class. This is as it should be.)

"How many integral units in your number?"

"How many fractional units?"

"Show me the same *not* mixed."



"How many numbers have you now?"

("I have two numbers—one integral having three units, the other fractional having two units.")

"Show me another different mixed number."

"How many integral units in this number?"

"How many fractional units in it?"

"Show me the same mixed."

"How many numbers have you now?"

("I have two numbers—one fractional with —— units, the other integral with —— units.")

"Show me two mixed numbers."

"Show me two integral numbers."

"Show me two fractional numbers."

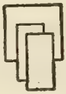
"Show me two numbers—one mixed, the other fractional."

(These dictations illustrate the form of the work. A dozen or more dictations as varied as possible, and the pupils are ready for (2).)

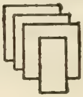
Note 1.—It may occasionally happen that a child will put two kinds of fractional units into a mixed number. For instance, if he is constructing such a number with two integral and two fractional units the latter might not be both halves, both thirds, or both fourths, etc. The teacher should be on the watch constantly for such constructions to prevent them. No warnings or explanations are advisable. The teacher simply makes the change quietly in each individual case. The development of what is *not* a mixed number is taken up in (c) below.

(2) Objects in the teacher's hands.

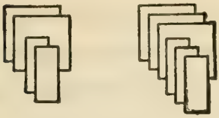
The teacher constructs and holds the constructions before the class, the pupils state orally what the constructions are.



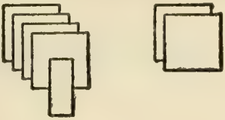
("You are showing me a mixed number having one integral unit and two fractional units.")



("You are showing me a mixed number having three integral units and one fractional unit.")



("You are showing me two mixed numbers—one having two integral units and two fractional units, the other three integral units and three fractional units.")



("You are showing me two numbers—one a mixed number with four integral units and one fractional unit, the other an integral number with two units.")

(These show the form of the exercises. Eight or ten such dictations will be sufficient before passing to (c).)

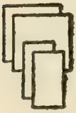
(c) Impossible Numbers

(1) Aim.—To develop what is *not* a mixed number—that large fractional units and small fractional units cannot be put into one group and that such a combination is not only not a *mixed* number but an *impossible* number. The term "impossible" is to be applied to groups having mixed fractional units. The development aims to bring to the child's attention that

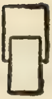
while he may "mix" wholes and fractions the "mixture" must not have *two kinds* of fractions. A large fraction with a smaller as a half with an eighth or a third with a fourth is not a mixed number. The child's natural inference is that there would be no limitations in the "mixing."

(2). THE DEVELOPMENT.

The teacher constructs and questions as follows:



"Is this a mixed number?" ("Yes.")



(These units are both fractional.)
"Is this a mixed number?" ("Yes.")
"No, this is *not* a mixed number." "It is an impossible number."

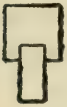
"People cannot put two kinds of fractions into one group."

"You must never let anyone catch you with two kinds of fractional units in one group—*never, never, never*. You would be laughed at. Such a number is impossible."

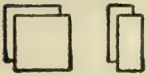
NOTE 2.—This principle must be very strongly impressed upon the pupils because it is fundamental. We reduce to a common denominator in addition and, in many cases, in subtraction for no other reason than here stated. This principle is the basis of addition development in Chapter XVIII. The teacher must bring it up very frequently in the weeks to come—with objects in her own hands perhaps—until it becomes established in the child's mind and he is quick to discover its violation.

(3) DEFINITION OF "IMPOSSIBLE NUMBERS"

(*c*) The pupils should now be taught to describe "impossible numbers."—An impossible number is one having two kinds of fractional units.

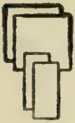


"Is this a mixed number?" ("Yes.")

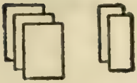


"What is this?" ("It is two numbers—an integral and a fractional number.")

(This represents a combination of the group of the previous dictation.)

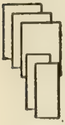


"What is this?" ("It is a mixed number—a number having two integral and two fractional units.")



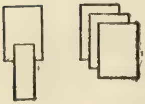
"What is this?" (It is two fractional numbers.")

(The combination of the two groups of the last dictation—unequal fractional units.)



"What is this?" ("It is *nothing*. It is *impossible* because it is a group having more than one kind of fractional units.")

(These objects are all fractions.)



"What is this?" ("It is two groups—one an impossible number because it has more than one kind of fractional units, the other a fractional number having three units.")

(These exercises illustrate the form that the dictations should take. Many (fifteen to twenty) exercises should be given with as great variety in form as possible.)

D. Definitions of Integer, Integral Number, and Mixed Number

The child should now be given definitions (formal) of integer, integral number, and mixed number.

1. Integer

"What is an integer?" ("An integer is a whole number.")

"Show me an integer."

"How many units in yours?" "In yours?" "In yours?"

"May an integer have one unit?"

"Show me such an integer."

"May an integer have more than one unit?"

"Show me such an integer."

"Will you" (to some child) "please tell me now what an integer may have? I want you to tell it all, and I want you to show me with objects while you tell it."

("An integer may have one unit" [here the pupil must hold up *one*]; "or, it may have more than one." [Here the child, still holding up the *one*, holds up in the other hand a group unit—two or three or other group.])

The teacher now asks another pupil the same question and must require the child to hold up the *one* and the *more than one* in the same way—

("An integer may have *one*" [holding up a one] "or *more than one*" [holding up a group of two or three] "unit.")

Several other pupils are asked one after another the same question and are required to answer in the same manner.

"I want you now" (to some pupil) "to tell the whole story about the integer—telling me first what it *is*, then what it *may have*."

("An integer is a whole number. It may have one unit or more than one unit.")

(The teacher must insist now and for many weeks to come that in *every* oral statement of this definition the child shall show the objects (the "*one*" and the "*more than one*"). He must have access to the objects whenever called upon for it. This object-action part in the definition is important for imagery development.)

The full statement as above should now be given by other pupils.

This is the child's definition of *integer*.

2. *Integral Number*

"What is an integral number?" ("An integral number is a whole number.")

"Show me an integral number."

"May an integral number have one unit?"

"May it have more than one unit?"

"I want you now" (to some pupil) "to tell me the whole story of what an integral number may have."

("An integral number may have *one* unit or *more than one unit*.")

(It should not be necessary to say that in giving this "whole story" the child must hold up, as before, the *one* and the *more than one* as he says "one" and "more than one.")

Others should now tell what it "may have" in the same manner.

"I want you now" (to some pupil) to tell me the whole story about the integral number—telling me first what it *is* and then what it *may have.*

("An integral number is a whole number. It may have *one* unit or *more than one unit.*")

Several others in turn should be called upon to define it in the same way.

This is the child's definition of *integral number.*

3. *Mixed Number*

"Show me a mixed number."

"How many integral units has yours?"

"How many fractional units?"

"May a mixed number have one integral unit?"

"May it have more than one integral unit?"

"May it have one fractional unit?"

"May it have more than one fractional unit?"

"I want you" (to some pupil) "to tell me all about what a mixed number may have."

("A mixed number may have one or more than one integral unit and one or more than one fractional unit of one kind.")

Other pupils, one after another, should be asked to give the statement of what a mixed number *has.* In stating what it has, the child should hold up the objects—first the *one* and then the *more than one* integral units, using both hands, then the *one* and the *more than one* fractional units, using both hands. In holding up the objects in this way the child must, of course, lay aside the integral objects and then show the "one" and the "more than one" fractional objects.

The formal definition of mixed number should now be formulated for the child to state and memorize—

"A mixed number is a number having one or more integral units and one or more fractional units of *one kind*."

These definitions should be called for with some degree of regularity—daily when possible—for many weeks.

E. Numbers Classified as to Form

It will be a very simple matter at this point to draw out a classification of numbers. It is the work of but one moment because an easy inference from a single question—"What kinds of numbers have we now learned?"

Nearly every child will make the inference at once and reply, "Integral numbers, fractional numbers, and mixed numbers."

The question should also take the forms, What are the classes of numbers? How many classes of numbers have we learned? The questions, as above, covering this classification of numbers should be frequently asked throughout the work in fractions.

SUMMARY.

1. Integer put into the number vocabulary. (*A*.)
2. Integral number must be freely used in class work interchangeable with integer (*B*.)
3. Mixed number developed—the *objective* mixed number not the written. (*C*, *a* and *b*.)

   "Impossible number" is introduced to describe a mixed number (objective) in which the child has used two kinds of fractions. (*C*, *c*.)

   "Impossible number" formally described. (*C*, *c* (3).)
4. Definitions and classifications.

   The formal definitions of integer, integral number, and mixed number developed. (*D*, *1*, *2*, and *3*.)

   The classification of numbers as integral, fractional, and mixed follows naturally here and is made by an easy inference. (*E*.)

Time required for the work of this chapter including reviews of Chapters II and III, three days. Total time for Chapters I and IV, nine days.

CHAPTER V

FRACTION PROOF AND THE DEFINITION—GENERAL

Topics:—Proving that a construction has the two necessary qualities. Development of the definition.

A. The Proof

Aim.—To prove fractional units by measuring them on a whole of the same kind and incidentally to develop the oral language of such proof—this language later to be the formal definition of "fraction."

1. Steps in the Proof

The child has collections of fractional units of various sizes accumulated by saving the pieces that he has made in his constructions—halves, thirds, fourths, etc. (although he, of course, does not know them by those names).

If we were to ask him to prove that any one of these pieces is a fraction it would be impossible for him to identify and collect the other parts of that particular "whole" to show what he had done. His only proof would lie in applying this piece with other equal pieces, if necessary, to a "whole" and showing by the fact that they exactly fit the "whole" that they are equal parts of a whole and therefore fractional units.

"Show me a fraction."

In answer to this dictation one child may show one or two or three or four halves; another, one or more thirds; another, one or more fourths; another, one or more fifths or sixths; etc. A child may show just what would make a whole or more than enough to make a whole. It is immaterial for development purposes what number of fractional units is shown.

"How many units in your fractions?" "In yours?" "In yours?"

"Prove that yours is a fraction."

Note.—It is very important that the proof be carefully made because it is to be the basis of the *definition* of "fraction" in the next section. The aim in developing this proof is to put meaning into the definition.

To make the development steps clear to the teacher we will suppose that four pupils, A, B, C, and D, have "shown" respectively in response to the above direction ("Show me a fraction") two of the six equal parts, five of the eight equal parts, one of the two equal parts, and four of the three equal parts of a whole, and that the teacher's development work as outlined below (in *B*) makes use of these constructions in order.

The teacher must understand that in this development she will be using at each step the construction made by some individual pupil—first A's, then B's, then C's, then D's. On this account there will naturally be a tendency to direct the instruction in each case to him personally. This must be guarded against. The instruction must be for the whole class. The teacher must have in each step the close attention of everyone.

To make full class attention possible it may be necessary, when thus considering a construction, to transfer the object work from the pupil's desk to the teacher's table so that the teacher's directions to the pupil, the latter's work of applying the fractional units to the whole, and his oral responses under the teacher's direction, may be easily followed by all.

This form of instruction (at the teacher's table) to three or four pupils in succession will suffice for the instruction of the whole class. Pupils will then begin to do the "applying" at their own desks and will give the oral work without help from the teacher.

2. *The Form of the Proof*

The teacher first considers A's construction.

The child is taught to take a whole and apply his pieces to it until it is covered, using in this case four other pieces of the same kind because there were but two in his construction.

He is then to state the result—"Six pieces like mine are just the same as the whole."

The child then gathers up the six pieces to compare them to show that they are equal—putting them together corner to corner and edge to edge. He then states the fact, "The pieces are equal." The conclusion is then stated, ("Therefore this" (showing again his original construction) "is a fraction.")

"Are the pieces in your fraction equal?" ("Yes.")

(This is a repetition because the child has just stated the equality. It is asked, however, in order to concentrate the attention again on this feature of the construction units.)

"How many like the six equal parts of a whole are there in your fraction?" ("Two.")

The teacher now considers B's construction.

The child applies his five pieces with three others of the same kind to a whole.

("Eight pieces like mine are just the same as the whole.")

He then gathers up the eight pieces and puts them together to prove that they are equal.

("The pieces are equal.")

He then shows his original construction.

("Therefore, this is a fraction.")

"Are the pieces in your fraction equal?" ("Yes.")

"How many like the eight equal parts of a whole are there in your fraction?" ("Five.")

The teacher now considers C's construction.

The child applies his pieces with another one like it to a whole.

("Two pieces like mine are just the same as the whole.")

He then gathers up his two pieces to prove their equality.

("The pieces are equal.")

He then shows his original construction—

("Therefore this is a fraction.")

"How many like the two equal parts of a whole are there in your fraction?" ("One.")

D's construction is now considered. The child applies three of his four pieces to a whole.

("Three pieces like mine are just the same as the whole.")

He then gathers up his four pieces to prove them equal.

("The pieces are equal.")

He then shows his original construction—

("Therefore, this is a fraction.")

"Are the pieces in your fraction equal?" ("Yes.")

"How many like the three equal parts of a whole are there in your fraction?" ("Four.")

Others are then required to prove their fractions in the same way with the same oral statements and the same questions at the close.

Constructions are afterwards made by the teacher for pupils to prove and give the oral statements in the same way, the teacher asking the same questions at the close.

B. The Definition Developed

1. The Fraction Make-up

a. What the Fraction May Have—Objective.

(1) Objects in the hands of pupils.

"Show me a fraction having *one* of the equal parts of a whole."

"Prove that it is a fraction."

"Show me a fraction having more than one of the equal parts of a whole."

"How many units in *your fraction?*" "In yours?"

"Prove that yours is a fraction."

(2) Objects in the hands of the teacher.

The teacher now constructs.

"Is this a fraction?" ("Yes.")

"How many of the equal parts of a whole in it?" ("Three.")

"Is this a fraction?" ("Yes.")

"How many of the equal parts of the whole in it?" ("Five.")

(3) Objects in the hands of pupils.

Pupils now construct.

"Show me a fraction having a very small number of the equal parts of a whole in it."

"How many units in your fraction?" ("One.")

"Prove that it is a fraction."

"Show me a fraction having a large number of the equal parts of a whole in it."

"How many units in your fraction?" "In yours?" "In yours?"

Other dictations of the same kind should be given, asking at one time for the smallest number of units possible (the "one" of the definition to be given later) and at another time for a fraction having several units (the "more than one" of the definition).

Occasionally the teacher should ask for proof that the construction is a fraction.

b. What the Fraction May Have—Oral Statement.

"May a fraction have *one* of the equal parts of a whole?" ("Yes.")

"Show me such a fraction."

"May a fraction have more than one of the equal parts of a whole?" ("Yes.")

"Show me such a fraction."

"Will you please *tell* me this, showing me at the same time a fraction of each of the two kinds."

("A fraction may have one" [here the pupil shows a fraction having one unit] "or more than one" [showing a fraction of two or more units] "of the equal parts of a whole.")

The teacher should have several pupils tell the fact and at the same time show the fractions in the same way.

c. What the Fraction May Have—Written Statement.

Pupils are now required (first one and then another) to *write* on the blackboard what a fraction "may have," showing the "one" and the "more than one" objectively as he writes.

("A fraction may have one" [showing one fractional unit as when stating it orally] "or more than one" [showing a group as in the oral statement] "of the equal parts of a whole.")

The teacher asks, "What may a fraction have?"

The child writes, "A fraction may have one" (showing the object) "or more than one" (showing the group) "of the equal parts of a whole."

The same question is put to another pupil and then another for reply in the same manner.

2. *The Fraction Definition*

"Now I am going to ask the question in another way and I want you to write the answer very carefully."—"What *is* a fraction?"

("A fraction is one or more than one of the equal parts of a whole" [showing the objects as before].)

This question is asked again and again for written and then oral reply. This is his definition.

3. Definition of Fractional Number

There is no need for development or explanation. The term itself was originally (Chap. III, *C*) thrown abruptly into use and was an easy inference from "fraction."

After the definition for fraction (with its objective accompaniment) has been given several times the teacher's question instead of being constantly, "What is a fraction?" becomes frequently, "What is a fractional number?" The child gives the same "fraction" definition with the same showing of the objects in the "one" and the "more than one" part of it.

SUMMARY.

The child proves that his construction is a fraction:—

(1) By applying his object or objects (with other similar objects if necessary) to a whole to show that they are equal to the whole and may, therefore, be considered "*parts*" of the whole (*A*, 2.)

(2) By applying the objects to each other to show that they are equal—*equal parts* of a whole." (*A*, 2.)

The definition is developed:—

(1) By an oral statement (*B*, *1 b*) of what a fraction has, objectively.

(2) By a similar written statement. (*B*, *1 c*.)

The definition is formulated from these statements. (*B*, *2*.)

Fractional number takes the same definition by inference. (*B*, *3*.)

Time required for the work of this chapter, including reviews of Chapters II and III, five days. Total time for the work of Chapters II to V, fourteen days.

CHAPTER VI

EXPRESSING FRACTIONAL NUMBERS

Topics:—The Names of Fractional Units. The Written Fraction—Numerical Form

A. Oral and Written Fraction

AIM.—To teach the fraction names and how to express fractions numerically

Steps in the Presentation.

(1) Objects in the hands of the pupils.

"Show me a whole."

"I want you to make this into fractional units of a particular kind. There must be just three."

After this construction the teacher tells the pupils what to call the pieces—"These pieces are thirds."

"Show me *two* thirds." (The child shows the group of two thirds.")

"Is this a fraction?" ("Yes.")

"Why is it a fraction?" ("Because it is two of the equal parts of a whole.")

"This is the way to write two thirds."

(The teacher writes it ($\frac{2}{3}$) on the blackboard.)

"Show me *one* third."

"You" (to some pupil) "may write it on the blackboard."

NOTE 1.—It will not be necessary to make any explanation or to teach the child to write it. Having been told how to write *two thirds* the child infers how *one* third and *three* thirds are written. In fact he will write the fractions in the development work that follows.

"Is it a fraction?" (referring to the concrete expression).

"What is the name of your fraction?" ("Thirds.")

"What is the number of units in your fraction?" ("One.")

"Yes, it is one of the equal parts of a whole."

"Show me three thirds."

"Is it a fraction?" ("Yes, it is three of the equal parts, etc.")

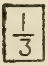
"What is the name of your fraction?" ("Thirds.")

"What is the number of units in your fraction?" ("Three.")

"Write it." ($\frac{3}{3}$)

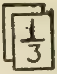
The teacher should now have these thirds marked and laid aside. On each piece the pupil should write its value ($\frac{1}{3}$) so that later when he has other fractions he can distinguish them readily as to name.

"You may now mark these pieces as thirds and we will lay them aside."



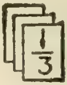
"Show me an integral unit."
"Make it into thirds."

"Show me *two thirds*."



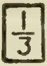
"Write it." ($\frac{2}{3}$.)

"Show me *three thirds*."



"Write it." ($\frac{3}{3}$.)

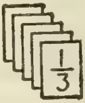
"Show me *one third*."



"Write it." ($\frac{1}{3}$.)

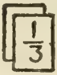
"Mark your pieces as before."

"Show me *five thirds*."



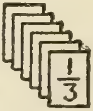
"Write it." ($\frac{5}{3}$.)

"Show me *two thirds*."



"Write it." ($\frac{2}{3}$.)

"Show me *six thirds*."

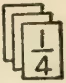


"Write it." ($\frac{6}{3}$.)

NOTE 2.—There are two questions that should be in quite regular use—(i) What is the name of your fraction? (ii) What is the number of units in your fraction?

"Show me an integral unit."
"You may make this into fourths."

Note 3.—It will not be necessary to suggest or explain. The pupils will infer easily from thirds what is meant, what to make, and how to make them.

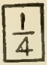
“Show me three fourths.” 

“Is this a fraction?” (“Yes, it is three of the equal parts, etc.”)

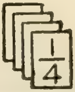
“Prove it.” (The proof is made in Chap. V, A 2, that (*a*) the pieces are together the same as a whole and (*b*) are equal.)

“Write it” (indicating some pupil) “on the blackboard.”

(The child writes “$\frac{3}{4}$.”)

“Show me one fourth.” 

“Write it.” ($\frac{1}{4}$.)

“Show me four fourths.” 

“Write it.” ($\frac{4}{4}$.)

Note 4.—Each of the fourths should be marked ($\frac{1}{4}$) and the pieces laid aside as was done with the thirds above—

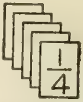
(The teacher must keep up the questions suggested above (1) as to the name of the fraction and (2) as to the number of units in the fraction.)

“Show me an integral unit.”

“You may make it into fourths.”

“You may mark each of the fourths.”

“Show me five fourths.”



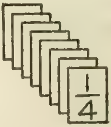
“Is it a fraction?” (“Yes, it is five of the equal parts of a whole.”)

“Prove it.”

(The proof is made as in Chapter V, *A 2*, that the four pieces are together equal to a whole and that the parts are equal.)

“You may write it on the blackboard.” ($\frac{5}{4}$).

“Show me eight fourths.”



“Is it a fraction?” (“Yes, it is eight of the equal parts of a whole.”)

“Prove it.”

(The child proves as before (*a*) that four of the parts are equal to a whole and (*b*) that the parts are equal.)

“Show me another integral unit.”

“You may make it into eighths.”

(No suggestion or assistance need be given. The inference from thirds and fourths as to what "eighths" means is perfectly easy and natural.)

"Show me five eighths."

"Prove that it is a fraction."

(The child proves as before that eight of the parts are equal to a whole and that the parts are equal—"It is five of the eight equal parts of a whole.")

"You may write it on the blackboard." ($\frac{5}{8}$.)

In the same way ask for seven eighths, three eighths, one eighth, eight eighths, etc. Have the expressions written. Ask now and then for proof that it is a fraction.

Mark and lay aside all of the pieces. The teacher's fraction objects which she makes and keeps on her desk for "objects in hands of the teacher" should be marked in the same way as those on the pupils' desks.

In the same manner make integral units into sixths, fifths, halves, ninths, and sevenths, marking the pieces in each case, and asking frequently (*a*) the name of the fraction and (*b*) the number of units in it.

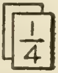
It is very important that all pieces be marked. There is no advantage in having a child puzzle over an object as to whether it is a fifth or a sixth, or whether it is a third or a fourth. The child's proper training is secured in making the fractions to order, not in recognizing the fraction value of an object at sight.

(2) Objects in the teacher's hands.

In this step the teacher makes the wholes into fractions—fourths, thirds, sixths, etc.—and shows the groups, the pupils to state what each group is and to write it. It will not be necessary to prepare the objects in the presence of the class. The teacher may prepare them beforehand ready for use. When she holds up the objects in each exercise she states how she made them so that it is not necessary that the pupil *see* them made.

EXERCISES

(i) "I made a whole into four equal pieces."

"What is this?" 

("It is two fourths.")

"What is the name of the fraction?" "What number of units has it?"

"Write it." ($\frac{2}{4}$.)

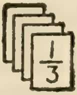
"Where did you write the name of the fraction?" ("I wrote it under the little fraction line.")

"Where did you write the number of units in your fraction?"

("I wrote it above the little fraction line.")

(ii) "I made each whole into three equal parts."

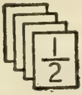
"What is this?"



("It is four thirds.")
"Write it." ($\frac{4}{3}$.)

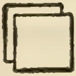
(iii) "I made each whole into two equal parts."

"What is this?"

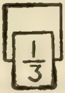


("It is four halves.")
"Write it."

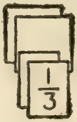
(iv) "What kind of a number is this?"
("It is an integral number.")



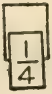
(v) "What kind of a number is this?"
("It is a mixed number.")



(vi) "I have two integral units and two thirds."
"Is this a mixed number?"
("Yes.")



(vii) "I have a half and a fourth."
"Is this mixed number?"
("No, it is an *impossible* number.")



NOTE 5.—This section gives an opportunity to review in this way integral, fractional, mixed, and impossible *numbers*. There should be several exercises of each of the kinds suggested above.

(3) Reviews.

Following or in connection with *B* there should be calls for definitions—

"What is a fraction?"

"What is an integral number?"

"What is a mixed number?"

"What is an integer?"

"What is a fractional number?"

"Where do you write the fraction name?"

"Where do you write the number of units in the fraction?"

B. Fraction Constructions

1. Fraction Constructions from Written Dictations

The pupil uses the fractional units that he accumulated and marked in the work under *A*, increasing his supply of each kind as dictations make additional objects necessary. When new fractional units are made they should be marked and put with the others of the same kind.

BLACKBOARD DICTATIONS

$\frac{3}{4}$	$\frac{2}{2}$	$\frac{7}{5}$	$\frac{2}{7}$	$\frac{7}{8}$
$\frac{2}{3}$	$\frac{5}{3}$	$\frac{4}{3}$	$\frac{9}{9}$	$\frac{7}{4}$
$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{7}$	$\frac{7}{4}$	$\frac{4}{3}$
$\frac{3}{6}$	$\frac{5}{9}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{3}{8}$

The teacher should dictate, as above, as many as the child can construct on his desk at one time.

After the pupils have completed their constructions and the teacher has examined them, the dictation should be erased and the pupils given pencils and paper to write the exercise from their concrete work. Two or three such exercises should be given before passing to the next section.

2. *Oral Statements of Construction Processes*

Aim.—To train the child to state orally the construction processes which he has been using in making fractions.

"How do you make fourths?"

("I take a whole and make it into four equal parts.")

Note.—The answer to this question is given in a *short form* full sentence. The answers to the two questions which follow are in full form full sentences. The teacher will choose the form she prefers her pupils to use; or, which would be better, she will preserve both forms by requiring at one time the "to make" and at another time the "I make" form.

"How do you make sixths?"

("To make sixths, I take a whole and make it into six equal parts.")

"How do you make thirds?"

("I make thirds by taking a whole and making it into three equal parts.")

SUMMARY.

The child is given the spoken name thirds—the name of the objects—and the written $\frac{1}{3}$. From these he infers the spoken and the written expressions for two thirds, three thirds, four thirds, etc. (*A*, (1).)

Exercises to give the pupil the opportunity to infer what is meant by fourths, sixths, halves, eighths, etc., and how to write the fractional numbers. (*A*, (1) and (2).)

Exercises in constructions from written dictations. This to test imagery. (*B*.)

Exercises in which the pupil is required to state orally how thirds, sixths, fourths, etc., are made. (*B*, 2.)

Time required for the development work of this chapter, including reviews of Chapter II–V, five days. Total time for the work of Chapters II–VI, nineteen days.

CHAPTER VII

FRACTION DEFINITIONS—SPECIAL

Topics:—The Definition of the Concrete Fraction—One Third,
Three Fourths, etc. The Definition of the Abstract
Fraction—Thirds, Fourths, etc.

A. Defining Fourths, Thirds, Sixths, etc.—Concrete

AIM.—To teach the child to define the *concrete* fraction and to give the definition in either of the two forms—the "is" form or the "when we" form.

The Two Definition Forms Developed.

(1) The "Is" Form.

"What do you mean by one third?"

("One third *is* one of the three equal parts of a whole.")

"Which is the name and which tells the number of units in the fraction?"

("*Third* is the name and *one* tells the number of units in the fraction.")

"What is two thirds?"

("Two thirds is two of the three equal parts of a whole.")

"Which is the name and which tells the number of units in the fraction?"

"What is six fourths?"

("Six fourths is six of the four equal, etc.")

"What does the *six* tell?" "The fourths?"

(2) The "when we" form.

"What do you mean by eighths?"

("When we make a whole into eight equal parts, one of these parts is called one eighth.")

"What is five sixths?"

("When we make a whole into six equal parts, five of these parts is called five-sixths.")

"What does "five" tell?" "What does "sixths" tell?"

NOTE.—The two forms in which these definitions should be given are indicated in (1) and (2). The child should be able to give a definition in either form—now in one, now in the other. "What do you mean by two sixths?" "You may tell me in the 'when we' form;" or, "You may tell me in the 'is' form."

B. Defining Fourths, Thirds, Sixths, etc.—Abstract

AIM.—To define the *abstract* fourth, third, sixth, etc. In *A* are outlined the *special* definitions—those for *one* fourth, *five* eighths, *seven* fifths, etc. We may speak of the fractions thus defined as *concrete* because they are quantitatively defined. The *abstract* fraction forms—fourths, sixths, thirds, etc., must also be defined. This section has to do with these definitions.

THE QUESTION AND THE DEFINITION FORMS

"What do you mean by fourths?"

("*When we* make a whole into four equal parts, these parts are called fourths.")

"What do you mean by thirds?"

("When a whole *is* made into three equal parts, these parts are called thirds.")

"What are sixths?"

("When a whole *is* made into six equal parts, these parts are called sixths.")

NOTE.—The teacher will note that this definition is also suggested in two forms, the "when we" form and the "is" form. The "when we" form becomes "when we make," etc. The two forms should be taught and required.

C. Definitions—Reviews

This section is to call the teacher's attention to the need of the definitions and oral statements that have been developed. The teacher must not permit a statement or a definition to lapse through disuse. If it was worth while to develop it, it is worth preserving.

In *all* reviews the statements and the definitions must be given in their original forms. If objects were used in the completed form originally, they must be used in the reviews.

"What is an integer?" (Chap. IV, *D 1*.)

"What is a fraction?" (Chap. V, *B 2*.)

"What is an integral number"? (Chap. IV, *D 2*.)

"What is a fractional number?" (Chap. V, *B 3*.)

"What is a mixed number?" (Chap. IV, *D 3*.)

"What is meant by one fourth?" "Two thirds?" etc. (See *A*, (1) and (2).)

"What are fourths?" "Sixths?" etc. (See *B*.)

"Where do we write the name of the fraction?"

"Where do we write the number of units in the fraction?"

SUMMARY.

The definition of the concrete fraction one fourth, three fifths, etc., in the "is" form (one fourth is, etc.) and also in the "when we" form (when we make a whole, etc.). (*A*, (1) and (2).)

Exercises in which the pupil is required to state which parts of the fraction expressions indicate the name and which parts the number of units in the fraction. (*A*, (1) and (2).)

The definition of the abstract fraction (sixths, eighths, etc.) in the "is" and in the "when we" forms—when a whole *is* made, etc., and when we make a whole, etc. (*B*.)

Time required for the development work of this chapter, including reviews of Chapters II–VI, five days. Total time for the work of Chapters II–VII, twenty-four days.

CHAPTER VIII

CONSTRUCTION IN EXPRESSIONS IN +, −, AND ×

Topic:—The Construction Forms of "Number Fact" Development Work Reviewed

Note 1.—It is absolutely necessary for the pupils to be familiar with the forms of concrete expressions in +, −, and ×, of integral numbers as outlined on pp. 71, 72, 77, 78, 86, and 101 of Vol. I.

They should be able to respond objectively to dictations like the following:

			+	×
5+2	2×3	6−2	4=	8=
3+6	3×4	8−5	6=	12=
6+1	4×5	7−4	10=	15=

If they have forgotten or have never learned how to make the constructions, the teacher must step aside from the regular line of this development to review or teach it. A very few exercises of ten minutes each will suffice. The outline in this section assumes that the pupil is able to construct such dictations and to read and write the constructions.

Aim. —To give the child the opportunity to infer directly that concrete expressions in fractions are exactly like corresponding expressions in integers. It should not be taught or explained.

A. The Constructions

No formal development is required. The teacher writes an exercise on the blackboard and the child constructs.

It will be necessary, of course, to use the paper objects and these require considerable space on the pupil's desk compared with the space needed for pegs or other small objects.

THE PRESENTATION

(1) The teacher writes on the blackboard an exercise of ten or twelve expressions according to the number that she thinks can be made on the child's desk at one time.

(2) The pupil constructs all of these expressions. (See Vol. I, pp. 72, 77, and 81.)

(3) The dictation is erased from the blackboard and the pupils with pencil and paper write their construction work.

(4) This completes the work. The papers are put back on the desk and another exercise is dictated, constructed, and written in the same manner.

BLACKBOARD DICTATIONS

			$+$	$\times$
$3+4$	3×4	$7-5=$	$9=$	$8=$
$5+3$	2×3	$\frac{5}{2}-\frac{3}{2}=$	$\frac{6}{3}=$	$\frac{10}{4}=$

$\frac{3}{2}+\frac{2}{3}$
$\frac{7}{3}+\frac{4}{3}$
$\frac{2}{5}+\frac{4}{5}$
$\frac{2}{9}+\frac{3}{9}$

$3\times\frac{2}{3}$
$6\times\frac{5}{6}$
$2\times\frac{3}{8}$
$5\times\frac{3}{2}$

$\frac{6}{9}-\frac{4}{9}=$
$\frac{7}{4}-\frac{5}{4}=$
$\frac{8}{5}-\frac{2}{5}=$
$\frac{7}{8}-\frac{4}{4}=$

$\frac{10}{5}=$
$\frac{9}{8}=$
$\frac{7}{4}=$
$\frac{6}{3}=$

$\frac{9}{7}=$
$\frac{12}{3}=$
$\frac{10}{9}=$
$\frac{6}{2}=$

The above indicate the forms of the dictations.

The number of exercises to be given under this section is left to the teacher's discretion. A total of 60 minutes would be a large amount of time.

CONSTRUCTION FORMS

On page 148 are the constructions of the first three expressions in each column of Blackboard Dictations above as they would appear on the pupils' desks.

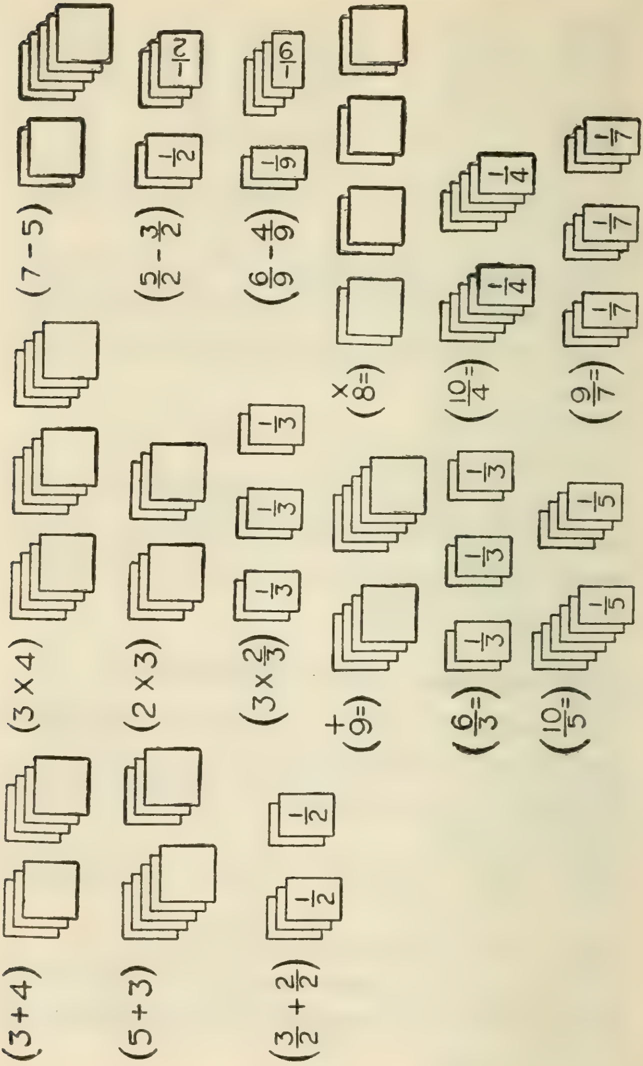
B. The Oral Work with the Constructions

There should also be oral work in connection with these constructions. The child should be taught to state what each kind of written expression tells.

3 + 4 "This tells me to take a three and a four."

5 × 2 "This tells me to take five twos;" or, "This tells me to take two five times."

6 − 3 = "This tells me to take six and take away three."


(3 + 4)
(3 × 4)
(7 − 5)
(5 + 3)
(2 × 3)
(5/2 − 3/2)
1/2
(3/2 + 2/2)
1/2
(3 × 2/3)
1/3
(6/9 − 4/9)
1/9
+ (9=)
× (8=)
(6/3=)
1/3
(10/4=)
1/4
(10/5=)
1/5
(9/7=)
1/7

$+$

$\frac{9}{3} =$ "This tells me to take nine thirds and make an 'and' expression."

$\times$

$\frac{8}{2} =$ "This tells me to take eight halves and make a 'times' expression."

$\frac{3}{2} + \frac{2}{5}$ "This tells me to take three fifths and two fifths."

NOTE 2.—It is of *utmost importance* for future development work in fractions that this oral work be introduced and that it be kept fresh and strong by frequent reviews. A few minutes every day or two should be given to it. Failure to recognize its importance will mean failure to hold the child at critical points in future development steps in fractions. The teacher will note that in each expression in "times" (5×2, $3 \times \frac{4}{3}$, $6 \times \frac{5}{8}$) there are two forms in which what it "tells" may be stated—five twos and two five times; three groups of four thirds, and four thirds three times; and six groups of five eighths, and five eighths six times. Both of these forms should be required. The two forms will be spoken of as reading forwards and reading backwards.

SUMMARY.

The aim—to revive the *integer* construction forms of the work in primary number and to continue it with *fractional* numbers.

Exercises in constructions without solutions (without =) and in the "will make" work of primary number. This is for the revival of the construction forms. (*A* and Note 1.)

Exercises in oral work with the constructions. (*B* and Note 2.)

Time required for the development work of this chapter, including reviews of Chapters II–VII, five days. Total time for the work of Chapters II–VIII, twenty-nine days.

CHAPTER IX

THE SIGNIFICANCE OF THE EQUALITY SIGN (=)

Topic:—That the Sign = After an Addition Partition, or Multiplication Expression Indicates that the Groups After Construction Must be "Put Together"

(1) Aim.—To make the child's knowledge of the significance of = in "+," "×," and "of" work more vivid—to teach him to know and to state the difference between $3+4$ and $3+4=$, between 2×6 and $2\times6=$, between $\frac{2}{3}$ of 6 and $\frac{2}{3}$ of $6=$. It will be remembered that in Grade I. objects were taken away before "reversed language" work was introduced (See Vol. I p. 147, Note.) Objective work was not needed—in fact, it would have been out of place in that stage of the child's number training. In fractions, however, the case is different. We are dealing with a subject in which we are concerned less about number "facts" than about number processes. This makes objective work in "reversed language" in fractions of utmost importance.

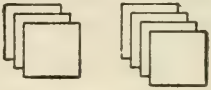
It will be necessary therefore, to give a few minutes to the development of this form of construction work because the child has had no previous experience with it.

(2) Method of presentation.

(*a*) *Expressions in* +

"Show me three and four"—at the same time writing it on the blackboard, 3 + 4.

The pupils make the groups—

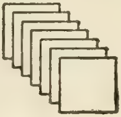


"What does the expression say when I write this after it?"—(the teacher writing " = " after the expression, 3 + 4 =).

"I will tell you. The = after it tells us to put the groups together."

"You may do with your objects what it says."

(The pupils now put the groups together.)



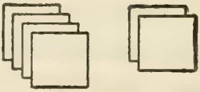
"How many in your group?" ("Seven.")

"4 + 2" (Written on the board.)

"What does it tell you?"

"It tells me to make a four and a two.")

"You may make the expression."



$4+2=$. (The teacher writes the " = " as given.)

"What does it tell you now?"

("It tells me to put them together.")

"You may do what it says.

"How many in your group?" ("Six.")

(b) *Expressions in* ×

"3×5."

"What does it tell you?"

("It tells me to make three fives.")

"Tell it to me backwards."

("It tells me to make five three times.")

"You may do what it says."

"$3\times 5=$."

"Now what does it say?"

("It tells me to put the groups together.")

"You may do what it says."

"How many in your group?" ("Fifteen.")

"$4\times 3=$."

"What does this tell you?"

("It tells me to make three four times and put the groups together.")

"You may do what it says."

"How many in your group?" ("Twelve.")

(*c*) *Mixed Exercises—Expressions in* + *and* ×

"$5+2=$."

"This tells me to make a five and a two and put them together."

(The pupil should be required to "make" and "put together" while talking so that when he finishes what it tells, he has the concrete expression completed.)

"$6\times 2=$."

("This tells me to make six twos and put them together.")

"You may read backward what it says."

("It tells me to make two six times and then put them together.")

"What does the = say?"

("It says, Put the groups together.")

The teacher should give several exercises—some in +, others in ×—for pupils to give full statements of what the dictations "tell." When the dictation is in full (with the =) the child should "make" and "put together" while he is making his oral statement, so that when the *statement* is *completed* the *construction is completed*. He should be required to state × expressions forwards and backwards. Included in these exercises should be expressions in fractions—$\frac{4}{3}+\frac{2}{3}=$, $3\times\frac{5}{6}=$, $4\times\frac{3}{8}=$, $\frac{5}{8}+\frac{6}{8}=$, $2\times\frac{5}{9}=$, etc.

SUMMARY.

Exercises for training the pupil to respond to the sign = after an addition, multiplication, or partition written expression by putting together into one group the used or "seen" groups after constructions.

Time required for the work of this chapter, including reviews of Chapters II–VIII, three days. Total time for the work of Chapters II–IX, thirty-two days.

CHAPTER X

PARTITION—CONSTRUCTIONS AND LANGUAGE

Topics:—Partition Constructions With Fractional Numbers—Even Partition. The Identity of the Partition Processes With Integral Quantities and With Fractions

Aim.—To give the child an opportunity through construction experiences to make the easy inference that partition of integral numbers and partition of fractional numbers are identical processes.

Note.—It is absolutely necessary that the pupils be fresh in the partition construction work in Grades II. and III. before undertaking this. (See Vol. I, pp. 197 and 227.

A. Constructions

(1) Form of blackboard dictations:

$\frac{3}{4}$ of 8 $\quad$ $\frac{7}{10}$ of $\frac{10}{9}$ $\quad$ $\frac{5}{6}$ of $\frac{9}{5}$ $\quad$ $\frac{2}{3}$ of $\frac{6}{5}$

$\frac{2}{3}$ of 9 $\quad$ $\frac{3}{8}$ of $\frac{16}{8}$ $\quad$ $\frac{2}{5}$ of $\frac{15}{2}$ $\quad$ $\frac{5}{6}$ of $\frac{12}{8}$

$\frac{4}{5}$ of $\frac{10}{2}$ $\quad$ $\frac{5}{9}$ of $\frac{18}{3}$ $\quad$ $\frac{3}{7}$ of $\frac{21}{4}$ $\quad$ $\frac{5}{8}$ of $\frac{16}{6}$

$\frac{1}{6}$ of $\frac{12}{4}$ $\quad$ $\frac{7}{8}$ of $\frac{24}{6}$ $\quad$ $\frac{5}{8}$ of $\frac{16}{7}$ $\quad$ $\frac{4}{7}$ of $\frac{14}{8}$

$\frac{2}{8}$ of $\frac{16}{7}$ $\quad$ $\frac{1}{2}$ of $\frac{12}{6}$ $\quad$ $\frac{8}{9}$ of $\frac{9}{8}$ $\quad$ $\frac{4}{6}$ of $\frac{12}{9}$

$\frac{5}{9}$ of $\frac{18}{2}$ $\quad$ $\frac{2}{3}$ of $\frac{6}{10}$ $\quad$ $\frac{8}{10}$ of $\frac{20}{3}$ $\quad$ $\frac{6}{7}$ of $\frac{21}{4}$

The above indicate the form of exercises under this section. No preliminary explanations or developments are necessary.

The number of expressions to be given for one seat construction exercise will depend upon the number of constructions that the child can put on his desk at one time.

The child should construct the exercise without aid or suggestion by the teacher. The teacher is free for other school room duties during the constructions.

2. Forms of constructions.

On page 158 are the forms of the constructions of the first two dictations in each column in (1) above.—(See Vol. I, pp. 195, 196.)

B. Oral Work with the Constructions

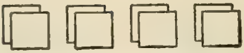
What was suggested in Chapter VIII, *B* about oral work with constructions applies here. The child should be given exercises in oral work with these partition constructions. Dictations should be given on the blackboard one by one for pupils to state what the expressions "tell" him.

$\frac{3}{4}$ of 12.

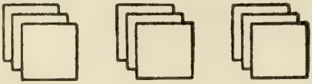
("This tells me to take 12, make them into four equal groups, show three, and hide one.")

The oral work and the construction should go together—the child taking the 12 as he *says* "take 12," making the four groups as he says "make them into 4 equal groups," etc.

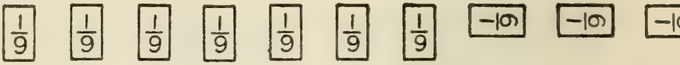
$\left(\frac{3}{4} \text{ of } 8\right)$



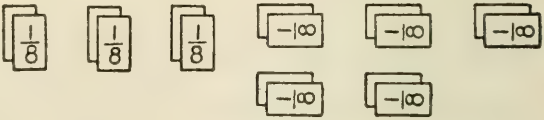
$\left(\frac{2}{3} \text{ of } 9\right)$



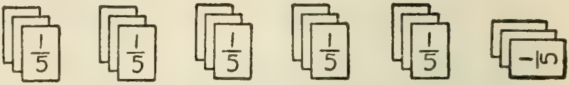
$\left(\frac{7}{10} \text{ of } \frac{10}{9}\right)$



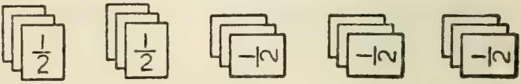
$\left(\frac{3}{8} \text{ of } \frac{16}{8}\right)$



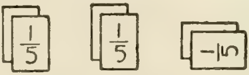
$\left(\frac{5}{6} \text{ of } \frac{18}{5}\right)$



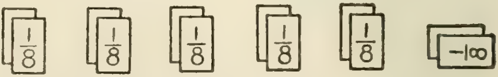
$\left(\frac{2}{5} \text{ of } \frac{15}{2}\right)$



$\left(\frac{2}{3} \text{ of } \frac{6}{5}\right)$



$\left(\frac{5}{6} \text{ of } \frac{12}{8}\right)$



$\frac{5}{6}$ of $\frac{18}{3}$.

("This tells me to take 18 thirds, make them into 6 equal groups, show five and hide one.")

The construction should go along with the oral statement as before—the child taking 18 thirds as he says "take 18 thirds," making the groups as he says "make them into 6 *equal* groups," etc.

NOTE.—Require the child, when he says, Make them into four equal groups; or, Make them into 6 equal groups or the like, to say it with heavy emphasis on the *equal*.

C. Construction and Oral Work with the Sign =

Dictations should be given with =. This the child will understand, without suggestions or explanations at this time, to mean just what it meant in Chapter IX—"put them together." It may be introduced somewhat abruptly as follows:

$\frac{4}{5}$ of $\frac{15}{2}$.

("This tells me to take 15 halves, make them into five equal groups, show 4 and hide one.")

(The oral work with the construction as in *B*.)

$\frac{2}{3}$ of $\frac{6}{8}$ =.

("This tells me to take 6 eighths, make them into 3 equal groups, show two, hide one, and then put them together.")

The oral work and the construction should go together as before. When the child says, "This tells me to take 6 eighths," he takes the 6 eighths. He makes the group while he is saying, "Make them into 3 equal groups." He turns the one group to the "hidden" position as he says, "Hide one." He puts the seven groups together into one group as he says, "Put them together."

The teacher will, of course, understand that "put them together" applies to the "seven" groups only. The hidden groups are put away in the general supply on the pupil's desk.

$\frac{3}{5}$ of $\frac{15}{3}$ = .

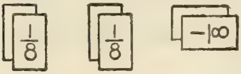
("This tells me to take 15 thirds, make them into 5 equal groups, show 3, hide 2, and then put the 'seen' groups together.")

The child suits the action to the word as he proceeds with his oral statement.

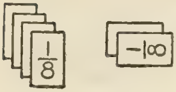
NOTE 1.—This oral work with constructions not only in partition but in addition, subtraction, and multiplication, must be kept up by frequent exercises throughout the whole fraction development period.

NOTE 2.—The object work for $\frac{2}{3}$ of $\frac{6}{8}$ = and $\frac{3}{5}$ of $\frac{15}{3}$ =, when completed, will be as follows:

(*a*) $\frac{2}{3}$ of $\frac{6}{8}$ =



Just before the child "puts them together."

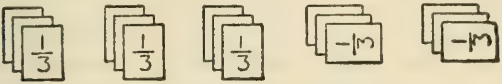


After he "puts them together." The hidden group may now be put aside with the papers in the child's general supply.

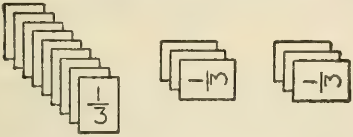
After completing the construction the child completes the dictation—$\frac{2}{3}$ of $\frac{6}{8}=\frac{4}{8}$.

(*b*) $\frac{3}{5}$ of $\frac{15}{3}=$

Just before he "puts them together."



Just after he "puts them together."



The hidden groups are now returned to the general supply.

After completing the construction the child completes the written expression—$\frac{3}{5}$ of $\frac{15}{3}=\frac{9}{3}$.

D. "Will Make" Work in Partition

The aim is to connect still more the concrete work in "of" with integers * and that with fractions.

There will be no need for explanations or introductory suggestions. The dictations will be given abruptly, the teacher relying for proper response in concrete work upon the pupil's power to infer. It is assumed that the child is fresh in the partition construction work in Grade II and III.

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* See Vol. I, p. 227.

DICTATIONS FOR CONSTRUCTIONS

$9 =$
$6 =$
$8 =$
$\frac{10}{2} =$
$\frac{15}{4} =$
$\frac{16}{6} =$
$\frac{12}{8} =$
$\frac{12}{9} =$
$\frac{14}{4} =$
$\frac{9}{7} =$
$\frac{8}{6} =$
$\frac{12}{3} =$
$\frac{18}{9} =$
$\frac{20}{8} =$
$\frac{16}{7} =$
$\frac{10}{8} =$
$\frac{6}{4} =$
$\frac{8}{9} =$
$\frac{12}{2} =$
$\frac{15}{10} =$

NOTE.—The pupil's oral work in exercises of this kind is the simple direct statement, This tells me to take—and make a partition expression.

SUMMARY.

Exercises in constructions without solutions with integral and with fractional dividends. These to develop the identity of the construction forms. (*A*.)

Oral work with constructions without solutions (without =). (*B*.)

Constructions and oral work with solutions (with =). (*C*.)

"Will make" work (dividends, to find and write the partition expressions) in partition. This to complete the development of the fact of the identity of integral and fractional partition. (*D*.)

Time required for the work of this chapter, including reviews of Chapters II–IX, five days. Total time for the work of Chapters II–X, thirty-seven days.

CHAPTER XI

DIVISION—CONSTRUCTION AND LANGUAGE

Topics:—Division Constructions With Fractions. Division Terms, Division, Divident, etc.

AIM.—The step from "has how many" constructions with integers to similar constructions with fractions in which no change in denominator is involved is very easy and natural, requiring no development work. The child simply needs the opportunity to construct. He will infer that division with fractions (objective) differs from that with integers only in the kind of objects used—that the processes are identical.

The aim of this section is to give the child the opportunity for such experience, to teach him, in connection with his work, certain terms—*division*, *divide by*, *quotient*, etc.,—and to express "part groups" in regular fraction language instead of in the "toward making another group of" language.

A. The Terms Division, Divide, and Divided By

1. The Oral Terms Developed

$8 \div 4 =$. (The teacher writes it on the blackboard.)

"What kind of an expression is this?"

("It is a 'has how many' expression.")

"Yes, but I want you to learn to call it also a division expression. We shall call this the short way."

"What does it tell you?"

("It tells me to take 8 and make it into groups of 4.")

"Yes, but there is another way to say it—a *short* way. It tells us to divide 8 by 4."

$12 \div 3 =$. (Written on the blackboard.)

"What kind of an expression is this?" "I want you to say it the short way."

("It is a division expression.")

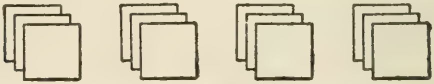
"Say it the long way."

("It is a 'has how many' expression.")

"What does it tell you (short way)?"

("It tells me to divide 12 by 3.")

"You may make it."



$\frac{10}{4} \div \frac{2}{4} =$. (Written on the blackboard.)

"What kind of an expression is this?" "Say it the long way."

("It is a 'has how many' expression.")

"Say it the short way."

("It is a division expression.")

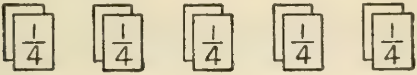
"What does it tell you (long way)?"

("It tells me to take ten fourths and make them into groups of two fourths.")

"Say it the short way."

("It tells me to divide $\frac{10}{4}$ by $\frac{2}{4}$.")

"You may make it."



The teacher should give a dozen or more additional exercises in the same manner asking the questions which have been suggested, with each:—

$\frac{16}{3} \div \frac{4}{3} =$
$\frac{12}{5} \div \frac{4}{5} =$
$\frac{18}{8} \div \frac{6}{8} =$

$\frac{14}{9} \div \frac{7}{9} =$
$\frac{10}{8} \div \frac{5}{8} =$
$18 \div 9 =$

$8 \div 2 =$
$\frac{15}{4} \div \frac{3}{4} =$
$\frac{12}{7} \div \frac{3}{7} =$

$\frac{20}{6} \div \frac{4}{6} =$
$9 \div 3 =$
$\frac{16}{5} \div \frac{8}{5} =$

2. *The Written Terms*

The following should be dictated orally for pupils to express in writing or to construct—

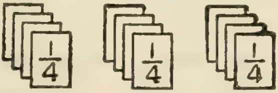
"Ten thirds has how many two thirds?"

(The pupils write it—$\frac{10}{3} \div \frac{2}{3} =$.)

"Divide twelve fourths by four fourths."

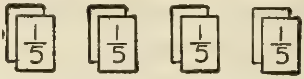
(The pupils write it.)

"Make it."



"Make what we mean when we tell any one to divide eight fifths by two fifths."

(The pupils construct.)



"I am thinking of dividing 12 by 6." "Write it."

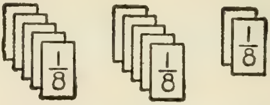
(The pupils write 12 ÷ 6 = .)

"Ten fifths divided by four fifths." "Write it."

Note.—This introduces "divided by" at a time when it will offer no difficulty to the child.

"Twelve eighths divided by five eighths."

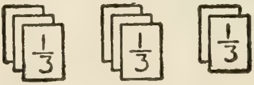
"Construct it."



B. Quotient and Reading the Part-group in Regular Fraction Language

1. *The Language Developed*

(*a*) "Make eight thirds divided by three thirds." (This to be dictated orally.)



"What you have made on your desks is called the quotient." "When we make a division expression, we make what is called the quotient of the expression."

"What does your quotient say?"

("It says two groups and two toward making another group of three pieces.")

"Can you think of a shorter way of reading the part-group?"

("Two thirds.")

"Write the whole expression with the quotient."

($2\frac{2}{3}$.)

"Show me the objective quotient."

(The child points to his construction.)

"Show me the written quotient."

(The child points to the written $2\frac{2}{3}$.)

"Does the part-group look like two thirds of a group?"

(*b*) "Nine fifths divided by four fifths."

"Write the expression."

($\frac{9}{5} \div \frac{4}{5} =$.)

"You may construct it."

"Read the quotient the *long way*."

("Two groups and one toward making another group of four pieces.")*

"Can you think of a shorter way to read it?"

---

* It is best to use the word piece or pieces in expressing the fractional groups when the units in the group are fractional. In $8 \div 3 = 2\frac{2}{3}$, we say, Two toward making another group of three." In $\frac{8}{5} \div \frac{3}{5} = 2\frac{2}{3}$, we say, Two toward making another group of three *pieces*. In the "short way" each is read two-thirds.

("Two and one-fourth groups.")

"Does the part-group look like one-fourth of a group?"

(*c*) "Divide eight by three." "Write it."

($8 \div 3 =$.)

"Show me the objective quotient."

"Write the expression including the quotient."

($8 \div 3 = 2\frac{2}{3}$.)

"Read the part-group the short way."

("Two thirds.")

"Does the objective part-group look like two thirds of a group?"

(*d*) "Divide eleven sixths by four sixths." "Write it."

($\frac{11}{6} \div \frac{4}{6} =$.)

"Make the objective quotient."

"Write the quotient." ($2\frac{3}{4}$.)

"Show me the objective quotient." (The pupil points to his construction.)

"What is it?" (Pupil reads it—$2\frac{3}{4}$.)

"Show me the written quotient."

"What is it, expressing the part-group like a fraction?"

("Two and three fourths groups.")

"Does the part-group look like three fourths of a group?"

2. *The Language Applied to Graphic Objective Work*

The teacher draws on the blackboard pictures of objective quotients for pupils to state or write the division statements. In making these graphic objective expressions, the teacher should state clearly as she makes each one, "These are halves;" or, "These are thirds;" or "These are eighths;" or, "These are integers;" etc. The pupil should be required to state the expressions in the "divide" or the "divided by" form and the part-groups in the *fraction* form.

(*a*) "These are fourths."

"This is a quotient." "What was the division expression?"

"I want the 'divided by' form."

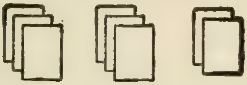
("It was eight fourths divided by three fourths.")

"State it in the 'divide' form."

("Divide eight fourths by three fourths.")

"State it in the long form."

("Eight fourths has how many groups of three fourths.")



"Read the quotient." ("Two groups and two pieces toward making another group of three pieces.")

"Read it in the fraction form."

("Two and two-thirds groups.")

"Does the part-group look like two-thirds of a group?"[*]

"Write the full division expression."

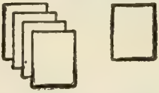
($\frac{8}{4} \div \frac{3}{4} = 2\frac{2}{3}$.)

(*b*) "These are halves."

"This is quotient." "Read it in fraction form."

"What was the division expression?"

("Divide five halves by four halves.")



"Tell me what it is in the divided by form."

("Five halves divided by four halves.")

"Give it the long way."

("Five halves has how many four halves?")

"Write the full division expression."

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[*] This question should be asked regularly. The aim in asking it again and again is to have the pupil acquire, if possible, the habit of looking at the *part-groups* to see that they *really are* what the fraction form of expression calls them. This habit is the development basis in "3 below." If the child has acquired this habit, that development plan will be successful.

($\frac{5}{2} \div \frac{4}{2} = 1\frac{1}{4}$.)

"Read the quotient the short way."

("One and one-fourth groups.")

"Does the part-group look like one-fourth of a group?"

The above suggestions indicate the forms of expressions that are to be worked out and the terms that should be emphasized in this section. Several dozen such expressions should be given before passing to (3). The *fraction form* of expressing the *part-groups* is of *utmost importance.* It will be used later in the further development of division of fractions, the more care and attention given it now the better prepared the child will be for that work.

3. *Written Quotients for Pupils to Construct and Write the Division Expressions*

(*a*) $3\frac{3}{4}$.

"This is a quotient." "Make the expression using thirds."

(Pupils construct.)

"Write the full division expression."

($\frac{15}{3} \div \frac{4}{3} = 3\frac{3}{4}$.)

"Does your part-group look like three-fourths of a group?"

(*b*) $5\frac{1}{2}$.

"This is a quotient." "Construct and write the full division expression using fifths."

(The pupils construct and then write the full expression:—$\frac{11}{5} \div \frac{4}{3} = 5\frac{1}{2}$.)

"Does your part-group look like one half of a group?"

(*c*) $6\frac{1}{4}$.

"This is a quotient." "The units in the division expression were thirds." "Construct and write the full division expression."

($\frac{25}{3} \div \frac{4}{3} = 6\frac{1}{4}$.)

"Does your part-group look like, etc?"

(*d*) $4\frac{3}{5}$.

"This is a quotient." "The units in the division expression were integral." "Construct and write the expression in full."

($23 \div 5 = 4\frac{3}{5}$.)

"Does your part-group look, etc.?"

Seat written work may be given in the following form—

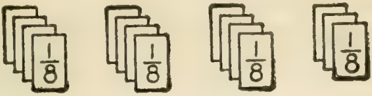
$2\frac{1}{2}$ (fourths)
$3\frac{1}{3}$ (eighths)
$4\frac{5}{6}$ (ninths)
$5\frac{2}{3}$ (integers)
$1\frac{2}{3}$ (fifths)
$3\frac{1}{4}$ (integers)

The word in the parenthesis tells the pupil the denomination to use in the division expression.

4. Oral Quotients for Construction of Division Expressions

(*a*) "The quotient is three and three-fourths. The objects are eighths. Construct and write the division expression."

The pupil constructs:—

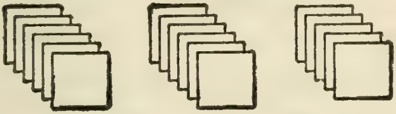


He then writes the full expression—($\frac{15}{8} \div \frac{4}{8} = 3\frac{3}{4}$.)

"Does your part-group look like three-fourths of a group?"

(*b*) "The quotient is two and five sixths. The objects are integers. Construct and write the division expression."

The pupil constructs:—



He then writes as before—($17 \div 6 = 2\frac{5}{6}$.)

The question is again asked about the part-group.

Several exercises of this kind—a dozen or more—should be given.

C. Reviews

Definitions of integer, fraction, mixed number, integral number, fractional number.

Where do we write the fraction name?

Where do we write the fraction number?

What do we mean by three fourths? (See Chapter VII, *A*.)

(1) The "is" form.

(2) The "when we" form.

What do we mean by fourths? (See Chapter VII, *B*.)

(1) The "is made" form.

(2) The "when we make" form.

Constructions every day in +, −, ×, and "of" with fractions, reviewing the oral, written, and construction language.

SUMMARY.

Division constructions using fractional numbers—the denominators common.

The terms divisor, divide, divided by. (*A*.)

Quotient—the term introduced. (*B*.)

Part-groups expressed orally and in written form in fraction language (*B*, (1) and (2).)

Finding the division expression from the written quotient. (*B*, 3.)

Find the division expressions from the quotient expressed orally. (*B*, 3.)

Time required for the work of this chapter, including reviews of Chapters II–X, five days. Total time for the work of Chapters II–XI, forty-two days.

CHAPTER XII

THE WRITTEN MIXED NUMBER

Topic:—To Discriminate Between $2\frac{1}{2}$, a Mixed Number, and $2+\frac{1}{2}$, a Number "*Not Mixed*"

AIM.—(1) To draw the child's attention to the difference in the forms of the construction between an expression that is a *mixed number* and the same (oral) expression when it is *not mixed.*

(2) To teach him how to write mixed numbers and how to change them to forms in which they are *not mixed.*

A. The Concrete and the Written Forms Developed

1. Objects in Pupil's Hands

"Show me a number having two integral and two fractional units—thirds."

"What kind of a number have you?"
("I have a mixed number.")
"How many numbers have you?" ("One.")
"What is your number?" *

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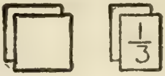
* The child should read it without suggestion or assistance. He is able to construct a mixed number

("Two and two thirds.")

"Show me the same not mixed." *

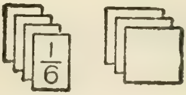


"How many numbers have you?"

"What are your numbers?"

("Two and two thirds.")

"Show me two numbers—one having four fractional units, sixths, the other three integral units." (The pupil will construct in the order that the dictation is given.)



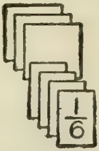
"How many numbers have you?"

("Two.")

"What are your numbers?")

("Four sixths and three.")

"Show me the same mixed."



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and to recognize one in the concrete. This development leads up to the *written* "mixed" and "not mixed" numbers.

* See to it that the pupils arrange their objects so that the *integer* shall stand *first*, the *fraction second* in order.

"How many numbers have you?"

("One.")

"What is your number?"

"Three and four sixths."

"Show me two numbers—one having three fractional units, halves, the other five integral units."

"How many numbers have you?"

("Two.")

"What are your numbers?"

("Five and three halves.")

"Show me the same mixed."

"How many numbers have you?"

("One.")

"What is your number?"

("Five and three halves.")

"Show me a mixed number having two integral units and four fifths."

"How many numbers?"

("One.")

"What is the number?"

("Two and four fifths.")

"Show me the same not mixed."

"How many numbers?"

("Two.")

"What are the numbers?"

("Two and four fifths.")

"Show me a mixed number of any kind you wish."

"How many numbers have you?"

"What is your number?" "Yours?" "Yours?"

"Show me the same not mixed."

"How many numbers have you?"

"What are your numbers?" "Yours?" "Yours?"

"Show me two and three-fourths mixed."

"Show me two and three-fourths not mixed.'

"Show me five and three halves not mixed."

"Show me five and three halves mixed."

(Several dictations should be given—twenty or more—the child being required to read each construction, to state how many numbers it contains, to read it, and to change "mixed" to "not mixed" and the reverse.)

2. *The Written Expressions*

"Show me three and three-eighths not mixed."

"How many numbers have you?"

"Two."

"Write the numbers of the blackboard."

($3 + \frac{3}{8}$.)

"Show me the same mixed."

"How many numbers have you?"

("One.")

"I will now show you how to write it so that people will know that the number is a mixed number." ($3\frac{3}{8}$.) "I have written the parts close together without the '*and*'." "This means that it is three and three eighths mixed."

"Show me three and one half not mixed."

"Write it on the blackboard."

($3+\frac{1}{2}$.)

"Show me the same mixed."

"Write it as a mixed number."

($3\frac{1}{2}$.)

"Show me four and two thirds mixed."

"Write it."

($4\frac{2}{3}$.)

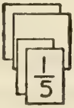
"Show me the same not mixed."

"Write the expression."

($4+\frac{2}{3}$.)

3. *Graphic Object Work for Pupils to Write*

The teacher draws objective expressions on the blackboard for pupils to state the kind of numbers (*mixed* numbers or *two numbers*) and to write them.

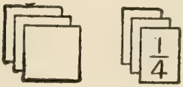


"What kind of a number?"

"Write it."

($2\frac{2}{5}$.)

"What is this?"



("Two numbers.")

"Write the numbers."

($3+\frac{3}{4}$.)

The teacher should give in this way several expressions, mixed numbers and numbers not mixed, for pupils to write.

4. *Dictations for Constructions*

The dictation exercises should be written on the blackboard for pupils to construct.

$2\frac{3}{4}$
$3\frac{1}{2}$
$2+\frac{4}{5}$
$5+\frac{1}{3}$
$3\frac{3}{2}$
$6\frac{1}{4}$
$5+\frac{2}{8}$
$6+\frac{5}{8}$
$5+\frac{5}{12}$
$6\frac{3}{9}$
$3\frac{3}{5}$
$8+\frac{2}{3}$
$9+\frac{3}{2}$
$7\frac{2}{3}$
$3+\frac{5}{6}$
$2+\frac{4}{5}$
$1\frac{3}{4}$
$4\frac{1}{3}$
$8\frac{7}{8}$
$4\frac{7}{9}$

After the construction work has been completed by all, the dictation should be erased and the pupils required to write the exercise from the concrete work on their desks. The teacher should examine the constructions on the desks and afterwards the written work from the constructions.

B. Classifications of Numbers as to Form

Pupils should give illustrations in written forms of the three kinds of numbers that they have learned—(See Chapter IV, *E.*)

(1) Integral.
(2) Fractional.
(3) Mixed.

These terms have been defined as follows:—

An integral number is a whole number.

A whole number is *one* or a group of *wholes*.

A fractional number is *one* or *more than one* of the equal parts of a whole.

A mixed number is a number having one or more integral units and one of more fractional units of one kind.

NOTE.—The teacher must bear in mind that all previous work must be kept fresh by occasional brief reviews—the construction and the language work of addition, subtraction, multiplication, division, and partition.

The arithmetical terms should be kept in use and the definitions that have been developed should be called for very frequently.

SUMMARY.

Exercises in the *construction* of "mixed" numbers—oral dictations—(See Chapter III) and in changing the form so that the construction will be "not mixed"—two numbers. (*A*, 1.)

Teaching to write such numbers so that the written expression will show that the construction would be a *mixed number*. (*A*, 2.)

Exercises in writing numbers from constructions so that the written expressions will indicate whether it is "mixed" or "not mixed." (*A*, 3.)

Numbers classified—integral, fractional, mixed. (*B.*)

Time required for the work of this chapter, including reviews of Chapters II–XI, five days. Total time for the work of Chapters II–XII, forty-seven days.

CHAPTER XIII

THE SERIES OF FRACTIONAL EQUIVALENTS—FROM INTEGERS

Topic:—The Construction of a Fraction Series that is Complete

AIM.—To give the child the construction experiences that will teach him that half is the largest possible fractional unit, third the next smaller, fourth the next smaller, etc.—that half, third, fourth, fifth, etc., constitute the fraction series complete, there being no fractional unit intermediate at any point in the series. It aims, in other words, to impress upon the child strongly the fact that when we have made a whole into two equal parts, another whole into three equal parts, another whole into four equal parts, etc., we have made all the fraction forms that are possible. This development is the introduction to fraction reductions.

A. The Constructions

1. The Object Work Developed

"Each of you may take a whole."

"I want you to make this into fractions but the fractions must be the very largest that can be made."

(There is no question that every one will make halves.)

"What did you" (to some pupil) "make?"

("Two halves.")

To another pupil, "What did you make?"

In answer to this question the teacher must bring out, "I tore my whole into two equal parts."

"You may take another whole."

"This time I want you to make your whole into the largest possible fractions excepting halves."

(Some pupils may make *fourths*. Most pupils, however, will respond more carefully and make *thirds*.

To some pupil, "What did you make?"

To another, "What did you make?"

Those who made fourths, will recognize that they have been "caught."

The teacher did not "catch" those who made thirds.

The question is again asked, "When I asked for the largest possible fractions excepting halves, what did you do?"

("I made my whole into three equal parts.")

The teacher should now have pupils take another whole and make it into the largest possible fractions. She should then ask as before, "What have you?" ("Halves.")

"What did you do?" (I made my whole into two equal parts.")

She should then have another whole taken and made into the largest possible fractions excepting halves. The questions, "What have you?" and "What did you do?" should be asked, as before, so as to bring out the answers, "I have three thirds," and "I made my whole into three equal parts.")

She should now have pupils take another whole and make it into the largest possible fractions excepting halves and thirds, the two questions being asked again and again so as to bring out that the pupils made the whole into four equal parts.

NOTE 1.—In all this work the fractions as made should be marked, each with its proper mark A ½ or A ¼ or B ½ or C ⅓ (see "Chapter I, *C*, 5.") and each piece placed in its proper section of the fraction receptacle (see "Chapter I, *C*, 7.) for use later in fraction constructions. In this way the fraction-parts now being made are not waste material.

The teacher will now repeat the work and the questions on "largest possible fractions" on "largest possible fractions excepting thirds," etc.

Following this will be similar work upon fifths, sixths, sevenths, eighths, etc. This work will bring out that:—

(1) The *largest possible* fractions are halves, the next largest are thirds, the next largest are fourths, etc.

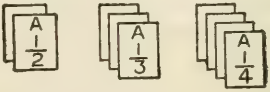
(2) The *largest possible* fractions are made by making a whole into two equal parts; the next largest, by making a whole into three equal parts; the next largest, by making a whole into four equal parts; etc.

2. *Seat Constructions*

The child should be given seat work in fraction equivalents:—

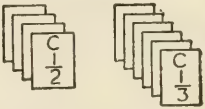
1 = 3 =
2 = 6 =
4 = 5 =

In 1 =, the child arranges the fraction series with objects across his desk (from left to right) in order as follows:—



etc.

In 2 =, he arranges the equivalents in the same way:—



etc.

NOTE 2.—There will not be room enough on a desk for more than two or three such series at one time.

3. *Writing Out the Series from the Seat Constructions*

After the object work has been arranged (two or three series like 1 =, 3 =, 2 =) the child should write the series:—

$1 = \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}$, etc.
$3 = \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5}$, etc.
$2 = \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{10}{5}$, etc.

B. Oral Statements on Constructions

1. *Statements on Single Constructions*

To make the work of *A* permanent the construction facts must be put into oral statements which the following questions suggest:

"How would you make the largest fractions?"

("I would take a whole and make it into two equal parts.")

"How would you make the fractions that are next smaller?"

("I would take a whole and make it into three equal parts.")

"How would you make the fractions that are next smaller?"

("I would take a whole and make it into four equal parts.")

The questioning should continue on the next smaller and the next smaller, etc.

2. *Statements on Series Constructions*

(*a*) "How would you make the largest possible fractions and then the others in order?"

("I would take a whole and make it into two equal parts, then another whole and make it into three equal parts, then another whole and make it into four equal parts, etc.")

(*b*) "If we should make a whole into the largest possible fractions in order what would we have?"

("We would have two halves, three thirds, four fourths, five fifths, etc.")

(*c*) "What are the fractions in order beginning with the largest?"

("Halves, thirds, fourths, fifths, etc.")

(*d*) "Is there a possible fraction between halves and thirds—smaller than a half and larger than a third?"

"Is there a possible fraction between thirds and fourths—smaller than a third and larger than a fourth?"

These questions suggest the lines of questions that should be asked.

(The aim here is to settle it definitely and firmly in the child's mind that halves, thirds, fourths, fifths, etc., are *all* the fractions there are—that no one can make an intermediate fractional unit in any part of the series.)

The teacher should ask these questions—(*a*), (*b*), (*c*), and (*d*)—frequently for weeks as recitation exercises. They involve information that the child should have and that he should be required to give very many times now in connection with object work as well as in the future when not using objects.

C. Seat Work in Equivalents—Without Objects

1. *Written Exercises*

Many exercises now and an occasional exercise in the future should be given in writing out fractional equivalents in series form. The child should do enough of this work to enable him to write series equivalents rapidly. In examining seat work the teacher must make sure that the written series is complete—no term overlooked. The term series must be put into use by the teacher. Teacher and pupil should make regular and free use of it.

FORM OF BLACKBOARD DICTATION

Write five terms of each series:—

1 = 4 = 6 =
5 = 3 = 2 =

2. *Oral Work*

The child must be asked in connection with this written work to state how he makes the fraction series from the various numbers. For illustration, he must state in answer to the question, How is the fraction series from 3 made, "I take three and make each unit into two equal pieces, then I take another three and make each into three equal pieces, then I take another three and make each into four equal pieces, etc."

He should be called upon for *oral* statements of equivalents in regular series—

"What will two make—the series?"

("Two will make four halves, six thirds, eight fourths, ten fifths, twelve sixths, etc.")

"What is the fraction series from four?"

("Eight halves, twelve thirds, sixteen fourths, etc.")

The pupils should be called upon frequently to give fraction series *orally*, to *write* series, to give the *names* of the fractions in series, etc.—

"Write the fraction series from two."

"What are the names of the fraction series from *one?*"

("Halves, thirds, fourths, fifths, etc.")

These questions and those suggested in *B* above should be in constant use now and should be brought up in review exercises frequently in the future.

3. The "Terms" of Series

The teacher must introduce the word "terms," teaching pupils to refer to the parts of series as "terms" of the series.

"What are the names of the fraction series to the fifth term?"

("The names of the fraction series to the fifth term are halves, thirds, fourths, fifths, sixths.")

"What are the first and fourth terms of the fraction series from two?"

("Four halves and ten fifths.")

"Write the first and sixth terms of the fraction series from three?"

($\frac{6}{2}$, $\frac{18}{6}$.)

SUMMARY.

Aim.—The construction of the largest possible fractional units, the next smaller in size, the next smaller, etc.

The Development.

- Object work.
  - The class work for constructions. (*A*, *1*.)
  - Seat work for constructions. (*A*, 2.)
  - Writing the series from the constructions. (*A*, *3*.)
  - The oral work on constructions. (*B*.)

Work without objects. (*C.*)
  Writing series. (*C, 1.*)
  Oral work on the written series. (*C, 2.*)
  The terms *series* (*C, 1*) and *terms* (*C, 3.*)

Time required for the work of this chapter, including reviews of Chapters II–XII, four days. Total time for the work of Chapters II–XIII, fifty-one days.

CHAPTER XIV

SERIES—PUNCTUATION AND VOCABULARY

Topics:—The Punctuation of Written Work. The Terms—Series, Denominator, Numerator, Equivalent

A. Punctuation of Written Work

In connection with the written language or composition work at this period the child is doubtless required to state the rule for and to use commas with series of words or phrases. The rule is, "When I have a series of three or more terms I must separate them by commas." This rule must be carefully observed by teacher and pupils in writing fraction series and, in fact, any kind of regular number series of three or more terms. The teacher should regard an exercise as seriously imperfect if the commas are omitted. The pupil should be required very frequently as in other forms of written language work, to state *why* the comma is used. He should point out the terms of the series, show that there are three or more, and show that commas separate them in accordance with the rule.

B. The Vocabulary

There are several terms, some of them already in use, that must at this time be established in the child's working vocabulary. They are "series," "terms," "odd," "even," "denominator," "numerator," and "equivalent." "Series" was introduced following the development of the fraction equivalents of 1, 4, 2, etc., as the name of the development *product*. Each set of constructions arranged in sequence was spoken of as a *series* and the numbers or groups which made up the series were spoken of as *terms*, thus bringing the words into use in connection with that which they signify. It remains for teacher and pupil to make constant use of them.

1. *Series*

a. APPLIED TO INTEGRAL NUMBERS

The development will rely upon the child's power to make an easy inference. He has used the term again and again in work with fraction series from integers. The plan of development is based upon the theory that the child should be able to infer from *series* and *integral* number what the series of integral numbers is.

METHOD OF PRESENTATION

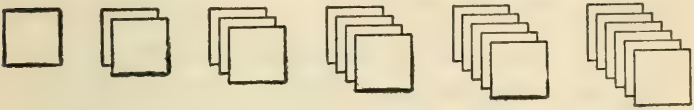
(1) Objects in the pupil's hands.

"Show me an integral number." *

"Objects aside."

"Show me a different integral number?" *

"Show me the full series of integral numbers to six terms."



"Objects aside."

"Show me the second, fifth, and eighth terms of the integral number series."

One or two more dictations will be sufficient except for an occasional exercise later.

(2) Objects in the teacher's hands.

The teacher should now reverse the form of exercise by making series for pupils to interpret—

("You are showing me the series of integers to three terms.")

---

* These two constructions are asked for in order to turn the pupils' lines of thought in the direction of integral numbers. They overcome the mental inertia and give the thought the proper direction. The next suggestion therefore, "Show me the series of integral numbers, etc.," finds the child in a state of mind to consider intelligently what such a suggestion should mean.

("You are showing me the first, fourth, sixth, and ninth terms of the integral number series.")

(3) Series orally without objects.

"What are the first eight terms of the series of integral numbers?"

"What are the third and tenth terms of the integral number series?"

b. THE "EVEN NUMBER" SERIES

In the *same manner* develop the series of even numbers—(1) Objects in the pupils' hands, (2) Objects in the teacher's hands, (3) Orally without objects.

(An even number is a number that will make even *groups* of twos.

An even number is one that may be *divided by* two without a fraction in the quotient.

An even number is a number that is *divisible by* two without a fraction in the quotient—without a fractional group in the quotient.

The word *divisible* is introduced in this connection. This and *divided by* should be used in class work constantly in order to give them places in the child's working vocabulary.)

c. THE "ODD NUMBER" SERIES

Odd numbers are taught in the *same manner.*

(An odd number is a number that will not make even *groups* of twos.

An odd number cannot be *divided by* two without a fraction in the quotient.

An odd number is a number that is not *divisible by*, etc.)

The child's attention should be called to the fact that even and odd are opposites—a number that is not even is odd, a number that is not odd is even.

2. *Denominator*

We used the word fraction *name* in Chapter VII, (A,) and many times in regular and review work since. The child should be given the term *denominator*. The word should be put into regular use instead of *name*.

("Denominator means fraction name.")

("The denominator of a fraction is the fraction name.")

"Where do we write the denominator?"

("We write the denominator under the fraction line.")

"What is the number under the fraction line?"

("Denominator.")

3. *Numerator*

The number of fractional units in given fractions has been asked for and given many times since it was first referred to in Chapter VII, *A*.

The term *numerator* should now be brought into regular use.

("Numerator means number of fractional units.")

("We write the numerator above the fraction line.")

("The numerator tells us how many units in the fraction.")

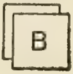
4. Equivalent

NOTE.—This term should be regarded as strictly interchangeable with "equal to" and its use should become as natural. There is practically nothing in arithmetic that compels the individual to discriminate between "equal" and "equivalent" in expressing quantitative equalities.

INTRODUCTION

The introduction is very simple.

(*a*) "Show me an integral number having two units."

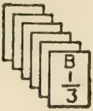


"Show me the same quantity in fractional units."

(Children will show $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, or other such number.)

"What is your fraction?" "Yours?" "Yours?"

"I want you" (to some pupil) "to write on the blackboard that your fractional number is equal to your integral number." (We will assume that the child thus addressed has $\frac{6}{3}$—



He may in response to this request begin to write the expression in words but the teacher must insist upon a statement in *number language*—$\frac{6}{3} = 2$—not in words.

The child writes $\frac{6}{3} = 2$. He is asked to read his statement—("Six thirds is equal to two"). One child after another is asked to write that his fraction is equal to his integer. Each reads his statement as above.

They are now taught "another way" to read this statement—six thirds is equivalent to two. Each statement is read in the "equivalent" language.

(*b*) "Show me three." (The pupils show it objectively.)

"Show me an equivalent fraction."

Different pupils will of course show different equivalents in response to this—$\frac{6}{2}$, $\frac{9}{3}$, $\frac{12}{4}$, or other number—objectively.

The teacher now asks one and then another to write on the board that his *fraction* is equivalent to his integral number. Each pupil as he writes his statement must read it as in

$\frac{6}{3}=2$ above. The form of the statement that his fraction is equivalent to his integral number requires that the fraction be made the first member of the equation and the integral number the second member.

(*c*) "Show me three fourths."

(The pupil constructs it.)

"Show me a fraction that is equivalent to it."

(Various fractions will be shown. Let us assume that one or more pupils will show $\frac{6}{8}$ objectively.)

"Write in number language what you have shown me."

(The teacher will observe that in this case the pupil was to construct an equivalent of the first named fraction, $\frac{3}{4}$. Following the language strictly the pupil must say that his $\frac{6}{8}$ is equivalent to the fraction. He must therefore write that the latter is equivalent to the former.)

The child must write it—$\frac{6}{8}=\frac{3}{4}$.

The teacher must keep in mind that the written statement must be strictly in accordance with the dictation in all cases. Otherwise we train the child to loose thinking. In this dictation the child was to show $\frac{3}{4}$ and afterwards a fraction equivalent to it. The written form as given above must be required.

Ask the pupils to read the statement in "equal to" language. Three or four similar dictations should be given. Have each statement read in "equal to" language also.

(*d*) "Show me two fractions that are equivalents."

(One and then another pupil is asked to state on the board in number language that his fractions are equivalents—($\frac{2}{3} = \frac{4}{6}$; or, $\frac{4}{5} = \frac{8}{10}$; or, $\frac{2}{2} = \frac{4}{4}$; or, $\frac{6}{9} = \frac{2}{3}$; etc.)

"Read what you have written."

("Two thirds and four sixths are equivalents.")

("Four fifths and eight tenths are equivalents.")

"Read yours" (to some pupil) "in the *equal to* form." "Yours." "Yours."

(*e*) Other "equivalent" forms should be taught in the same way.

$\frac{5}{7} = \frac{10}{14}$ read forwards is—$\frac{5}{7}$ is equivalent to $\frac{10}{14}$.

Read backwards it is—$\frac{10}{14}$ is the equivalent of $\frac{5}{7}$.

Read together, as in (*d*), it is $\frac{5}{7}$ and $\frac{10}{14}$ are equivalents; or, $\frac{10}{14}$ and $\frac{5}{7}$ are equivalents.

Very many exercises should be given for *forward*, *backward*, *together*, and *equal to* readings. Teacher and pupil must make every possible use of equivalent until its place in the working vocabulary is as natural as that of *equal to*.

SUMMARY.

- The use of commas to separate the terms of a written series of three or more terms. (*A*.)
- The vocabulary. (*B*.)
  - Series—(*B*, *1*.)
    - Applied to integral numbers. (*B*, *1 a*.)
    - The series of even numbers. (*B*, *1 b*.)
    - The series of odd numbers. (*B*, *1 c*.)
  - Denominator—introduced and defined. (*B*, *2*.)
  - Numerator—introduced and defined. (*B*, *3*.)
  - Equivalents—introduced and applied to the terms of fraction series. (*B*, *4*.)

Time required for the work of this chapter including reviews of Chapters II–XIII, five days. Total time for the work of Chapters II–XIV, fifty-six days.

CHAPTER XV

THE SERIES OF FRACTIONAL EQUIVALENTS—FROM FRACTIONS

Topics:—The Constructions of Full Series—From Fractions. From Mixed Numbers. The Oral Work With Series Forming

AIM.—To develop term by term the series of smaller fractions from halves, thirds, fourths, etc.; to construct the series from half, third, etc., without the omission of a term; to state orally what the construction processes for these series are; to write the full series in order from each fraction without objects; and to give each of the series in full orally without objects. This work is preparatory to common denominator, lowest common denominator, and lowest terms in later developments.

A. The "Half" Series Developed

1. The Terms Developed One by One

"Each of you may take a half."

"I want you to make your half into smaller fractional units but they must be as large as possible."

(Under this direction the child will, of course, make two fourths.)

"What did you make?" to some pupil.

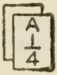
("I made two fourths.")

"How did you make these units?"

("I tore my half into two equal parts.")

(Others are asked in the same way what they made and how they made the units.)

"You may put these two fourths at the back of your desk at the left in group form."



"Each of you may now take another half."

"I want you to make this half into smaller fractional units. They must be just as large as possible but must not be fourths."

(Under this description the child must make three sixths.)

The teacher now asks, as before, of one and another pupil what he has made and how he made the units.

("I made three sixths.") ("I tore my half into three equal parts.")

"You may put your three sixths at the back of your desks next to the other group, on the right of it."

"Each of you may now take another half."

"I want you to make this half into smaller fractional units. They must be just as large as can possibly be made except that they must not be fourths or sixths."

(The pupils must, of course, construct four eighths.)

The teacher asks again what has been made and *how* the units were made.

The child's replies are, ("I made four eighths"), ("I tore my half into four equal parts.")

The group of four eighths is now placed at the back of the desk in the series.

In this way the teacher develops the "half" series to 12ths or 14ths or 16ths as she desires.

The pupils should now break up the series, restoring the fraction objects to their proper places in the fraction receptacles.

2. *Construction of the Full Series*

"Now I want you to take halves and make the smaller fractional units in order—the largest possible, then the next smaller, then the next smaller, and so on. I want to see if you can do it by yourselves without leaving out a term so far as you go."

While this work is being done the teacher goes about the room to see if any of the constructions are imperfect—a term of the series left out.

If, on inspection of the work on a desk, the teacher finds that a term has been omitted, she may show a pupil *where* the term should be but the pupil must find for himself what it is and put it in.

"What did you make? I want you to tell me what each term of your series is." (This to some pupil.)

The pupil responds orally putting his finger on the term of the objective series as he states it—"I made two fourths, three sixths, four eighths, five tenths, six twelfths, etc."

Another pupil, then another, then another should be asked this question, "What did you make?" and be required to give answer in the same way.

3. Oral Statements of Construction Processes

"Tell me how you made the terms of your series in order."

("I took a half and made it into two equal pieces, then another half and made it into three equal pieces, then another half and made it into four equal pieces, etc.")

This question should be put to several pupils and the full answer required as above.

This construction of the "half" series should be made by the pupils three or four times. The two questions as above—"What did you make?" and "How did you make the terms of your series in order?" should be asked of several pupils after each construction.

4. *Writing the Full Series Without Objects*

After the pupils have constructed the "half" series three or four times in this way, they should be asked to write the series without objects—

"$\frac{1}{2}=\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}$, etc."

5. *Oral Statements of the Series Without Objects*

After the pupils have written the series a few times without objects, as in 4, they should be required to give it orally.

Note.—The following questions should be a part of this oral work. Similar questions should be used in connection with all fraction series work:

How would you make the full series of fractions from one half?

What is the fraction series from one half to the fifth term?

What is the third term of the fraction series from one half?

What are the 2d, 4th, and 6th terms of the fraction series from one half?

What are the denominators of the first four terms of the fraction series from half?

What are the numerators of the 4th and 6th terms of the series from one half?

B. The "Thirds" Series

The teaching plan for the series from "thirds" is the same as that for the series from "half." The introductory steps for the development of the terms one by one, the construction of the full series without omission of a term, the statement of the construction processes, writing the full series without objects, and the oral statement of the series are the same for "thirds" as for "half." The oral work suggested in *A* 3 and 5 Note should be given with thirds (one third, two thirds, four thirds, etc.).

C. The "Fourths" and Other Series

The work outlined in *A* should be followed with fourths, fifths, sixths, etc., to twelfths excepting that actual object work for sevenths, ninths, tenths, elevenths, and twelfths, is not necessary. The child's imagery will be as clear without objective work on those fractions as with it. His power to reason—to infer—establishes the mental picture of series from fractions smaller than sixths or eighths without actual construction experience. The oral statements suggested in *A 3* and the questions in *A 5* Note should not be overlooked in the written and oral work with sevenths, ninths, etc. The written and oral work are the same as if objective work had been given.

Note.—The teacher will note what is said in (Chap. I C, 11) about "making" and "exchanging" fraction objects. When the child's accumulation of objects is large enough, in the teacher's judgment, for all reasonable construction uses in the future, the constructions should be made by "exchanging" rather than "making." In his oral work, then, the pupil instead of saying, "I tear" or "I make," says, "I exchange." In this way time is saved in the concrete work and there is no waste of paper for objects.

D. Seat Work in Series

1. *Constructions*

Exercises should be given in constructing series, the teacher to examine each desk to see that the constructions are right and that no term is omitted. In this work the dictations should specify the number of terms to be constructed in each case. (It is not necessary or advisable to dictate constructions that require fractional units smaller than 40ths or 42ds or 48ths.) Only a few exercises of this kind need be given at this time. An exercise should be given occasionally, however, for several weeks—one each week, perhaps.

Forms of Dictations—Written

Construct the series of equivalents from—

$\frac{2}{2}$ = (to fifth term)
$\frac{2}{3}$ = (to fourth term)
$\frac{3}{4}$ = (to six terms)
2 = (to five terms)

$\frac{2}{4}$ = (to seven terms)
$\frac{3}{8}$ = (the 2d, 4th and 5th terms)
3 = (2d, 4th, and 6th terms)
3 = (2d, 4th, and 6th terms)
$\frac{4}{8}$ = (1st, 3rd, 4th terms)
$\frac{3}{10}$ = (four terms)
$\frac{1}{5}$ = (4th and 5th terms)
$\frac{1}{8}$ = (four terms)
$\frac{5}{6}$ = (three terms)
$\frac{3}{7}$ = (four terms)

NOTE.—The term equivalent is in use from Chapter XIV, but in the pupil's work so far all of his series have *not* been series of *equivalent* terms. We must now impress upon him strongly that *every fraction series* is a series of equivalent terms.

The child may construct fraction series the terms of each of which are equal. This is due, not to *effort* at equality but to the construction *process*. There is nothing in the work of construction that makes the fact of this equality a part of the thought product. Constructions hold attention upon the equality of the *units of each whole* but not upon the equality of *terms in a series*.

The fact of this *equality of terms* is a very important part of the product of series work; and, attention to it not being secured indirectly through the constructions, it must be secured by direct means—by questions which recall the fact. For this purpose, the following question should be asked upon each of the constructions in this exercise and upon each series in all the series work, whether "with objects" or "written without objects," in (F) below and in Chapter XVII, Which is greater in quantity this term or this term (the teacher pointing to one construction and then to another in the series or to one and then to another *written* term if the work is "without objects"). The pupil should be required to express his answer in terms of equivalent—"They are equivalents." This question should be asked regularly, and two or three times upon each series.

2. *Writing Series—Without Objects*

Exercises like those in 1 above to be written without objects—

(*a*) From written dictations like those in 1.

(*b*) From oral dictations.

3. *Oral Statements*

In connection with this construction work and written work without objects the suggestions for oral work and the questions outlined in *A 3* and *A 5* Note above must not be neglected. The following, some of which are among those just referred to, are of particular importance in all fraction work:

(*a*) What did you make? (If the series is from *half*, the child's reply is, I made two fourths, three sixths, four eighths, etc.; if the series is from one third, the reply is, I made two sixths, three ninths, four twelfths, etc.)

(*b*) How did you make your series? (If the *half* series, the reply should be, I took a half and made it into two equal parts (or exchanged it for two fourths), then I took another half and made it into three equal parts (or exchanged it for three sixths), etc.; if the *thirds* series, the reply is in corresponding language.)

(*c*) When I make a (half, third, or fourth, etc.) into two equal parts what do I have? Into three equal parts? Into four equal parts?

(*d*) How would you make fourths from halves? Sixths from halves? Tenths? Eighths?

(Corresponding questions as to making fractional units from thirds, fourths, etc.)

(*e*) If you have a *half*, how would you make sixths? Fourths? Tenths? Eighths?

The reply is of course, I would make my half into three equal parts; or, I would make my half into two equal parts; etc.)

(*f*) What is the full fraction series from half?

(It is two fourths, three sixths, four eighths, etc.)

From one third? From two thirds? From one fourth? etc.

How would you make the series from two thirds?

(I would take two thirds and make each into two equal parts, then I would take another two thirds and make each into three equal parts, then I would take another two thirds and make each, etc.)

(*g*) Give me the denominators of the first four terms of the fractions in the series from half. From fifth. From seventh.

(*h*) What is the numerator of the 2d term of the series from $\frac{3}{8}$?

How would you make the series from three eighths? (I would take three eights and make each into two equal parts, then I would take another three eighths and make each, etc.)

What are the numerators of the first three terms of the series from $\frac{3}{7}$?

E. Finding the Number of a Term from the Construction

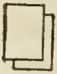
This is series work in reversed form. Heretofore the pupil's work has been to construct or write full or partial series or definite terms of series from given numbers. In this he is given the problem of finding the number of a term from the teacher's construction.

1. *Object Work*

Plan.—The objects used should be large enough so that the construction at the teacher's table may be seen by pupils in all parts of the room.

Instead of constructions with objects, the teacher might use graphic objects on the blackboard. Graphic object work for this section is more rapid than actual constructions and answers every purpose.

The teacher constructs—



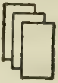
"This was made from a half." "What is it?"

("It is two fourths.")

"Which term of the series is it?"

("The second.")

Another construction is given—



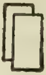
"This is from a third." "What is it?"

("It is three ninths.")

"Give the number of the term."

("It is the third term.")

"This is equivalent to one sixth."



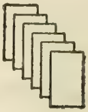
"What is it?"

("It is two twelfths.")

"What is the number of the term in the series?"

("It is the second.")

"This is from three fourths."



"What is it?"

("It is six eighths.")

"What term of the series?"

("The second.")

"This is equivalent to two."

"What is it?"

("It is four halves.")

"Number of the term?"

("The second.")

Very many exercises of this kind should be given at this time and frequent reviews later.

2. *Oral Work Without Objects*

This is *1* without objects. The pupil should respond orally unless the teacher thinks it best to have occasional exercises with written responses.

"I am thinking of an equivalent of $\frac{5}{6}$. It has 15 units. What is it?"

("It is fifteen eighteenths.")

"Which term of the series is it?"

("The third.")

"I am thinking of one of the equivalents of $\frac{2}{3}$. It has 8 units. What is it?"

("It is eight twelfths.")

"Which term of the series is it?"

("It is the fourth.")

F. Equivalent Series from Mixed Numbers

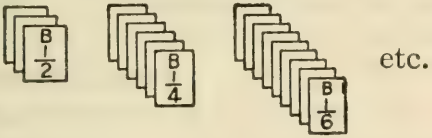
It has become fully established in the child's mind by the concrete, written, and oral exercises of this chapter that all series of fractional equivalents are constructed by making the unit (or each unit of the group) into two equal parts, then another into three equal parts, then another into four equal parts, etc.

What he will do when asked to make a series from a mixed number (as $1\frac{1}{2}$) can not be fully conjectured. He will do one of two things—exchange his whole for $\frac{2}{2}$ and then proceed to construct a series from $\frac{3}{2}$ as in his other series work; or, he will attempt first to exchange the 1 for $\frac{2}{2}$ and the $\frac{1}{2}$ for $\frac{2}{4}$ (thus literally making each of the units into two equal parts) and then to put the parts together, making an "impossible" mixed number. Some will follow the former plan and others the latter.

The former procedure is, of course, the proper one. One or more of the pupils will without doubt follow the latter procedure and, if so, their work will lead them to discover and correct their error; because, to follow the plan that was followed in all their series constructions (Chapter XIV), they would take the group, $1\frac{1}{2}$, and make each unit into two equal parts for their first construction, then another $1\frac{1}{2}$ and make each unit into three equal parts for the next term, etc., and would have each time what they have recognized over and over again as "impossible" numbers. Their very first construction would be an "impossibility" and they would at once recognize it as such.

It is not the teacher's business, it would be poor teaching, to correct the child's work or to "explain" the matter to him. It is one of the problems that the child should have the privilege of mastering for himself. The teacher should wait. The child must be permitted to work out the problem of getting the first construction and then the second construction from his mixed number into a form that will be "possible."

This is one of the steps in fraction development in which the teacher should "keep hands off" and leave the complete solution of the problem to the child alone. His construction will be finally in the following form:



The presentation.

1. Object Work

The teacher writes 1½ on the blackboard.

"Make the series of equivalents from one and one-half *mixed*."

This suggests all that there is to the teaching plan. The dictation is written, the request is made as above (to construct the series), and the pupils are held individually to the construction—each being left to his own resources to make an unbroken series that involves "possible" numbers.

After two or three others have been worked out, the pupils are ready for constructions independent of the teacher—work in which the child constructs several mixed number series on his desk and then writes them from his constructions.

Form of dictation.

Construct the equivalents to the fourth term—

$2\frac{1}{2}$ $1\frac{1}{3}$ $1\frac{5}{6}$ $3\frac{1}{9}$

$3\frac{2}{3}$ $2\frac{5}{8}$ $2\frac{1}{7}$ $2\frac{3}{4}$

The questions, "How did you make?" "What are the denominators?" "What is the numerator of the—term of the series from —?" "What is the — term of the series from —?" etc., must be in frequent use. The question also in D Note.

2. *Written Work Without Objects*

Dictations are written on the blackboard for pupils to write series to a given term. Other dictations are given for pupils to write certain terms as 1st, 3d, and 5th; 3d, 4th, and 7th; etc.

3. *Oral Work*

Following the exercises on written series 2 dictations oral and written should be given for mental work on series and oral statements.

The suggestions for questions and oral statements in other sections of this chapter apply here.

SUMMARY.

1. The aim—the preparation for the development later of common denominator, lowest terms, etc.
2. Exercises in series forming.
    - Object work.
        - The construction of the series from "half" term by term. (*A*, *1*.)
        - The construction of the full series. (*A*, *2*.)
        - The oral work (*A*, *3*.)
    - Work without objects.
        - Writing the full series in order. (*A*, *4*.)
        - The oral work and questions. (*A*, *5* and Note.)
3. The "thirds" and the "fourths" series. (*B*, and *C*.)
4. Exercises in general series forming.
    - Object work.
        - Series from miscellaneous fractions. (*D*, *1*.)
        - The term equivalent. (*D*), Note.)
        - The oral work on the series. (*D*, *3*.)

Work without objects.
  Writing series to a given term. (*D*, 2.)
  The oral work on series. (*D*, 3.)

5. Finding the number of the term. (*E*.)
   From constructions of single terms.
   Orally from oral statements or descriptions of terms.
6. Series from mixed numbers. (*F*.)
   The series constructed.
   Written series without objects.
   The oral work.

Time required for the work of this chapter, including reviews of Chapters II–XIV, ten days. Total time for the work of Chapters II–XV, sixty-six days.

CHAPTER XVI

THE NUMBER VALUE OF OBJECTS

Topics:—That Number Values of Objects are Arbitrary. How to Impress this Fact of Arbitrary Values. Teaching the Child How to Set and Test Values

AIM.—To teach that the number value of an object is artificial and entirely arbitrary, to impress upon the child strongly that he must so regard it, and to teach him how to find what its assigned value is in any case.

An individual may have a piece of paper (6″×6″, perhaps) and properly say of it, "Let us call this *one*," or "We will call this one *half*"; or, tearing it into any number of equal parts (four, perhaps) and taking one of the pieces thus made, say of it, "We will call this *one*—a *whole*." We do not hesitate to use a paper or a splint or other object in an illustration or a demonstration as *one*, a whole, regardless of the fact, perhaps, that it may be equal in all respects to another object that has been used in another discussion as a half, a fourth, or other fraction. We recognize the right of the individual thus arbitrarily, for purposes of his own, to set a number value on an object; and we accept this value without question in the consideration that we give to the matter under discussion.

The child somewhere in his fraction development work must have experiences that will bring these facts very forcibly to his attention. He must learn to change values arbitrarily—using an object at one time as a fractional unit, at another time as an integer.

He must learn too that an object in his own hands and under his own control is valued as he himself shall determine, and that a similar object in the possession and under the control of another person has the value that that possessor chooses to assign to it. He must learn, in other words, that the use of an object fixes its number value.

He must learn (1) that it is not safe to infer that a given object in the hands of another person, although it looks like one of his own halves or thirds or fourths, is on that account regarded by its possessor as a fraction of that name; and (2) that it is foolish and useless to venture an estimate of its number value until the possessor answers certain proper questions about his object—questions relating to such value.

DEVELOPMENT PLAN

The development has three steps—

A. Exercises in using objects in the child's hands with one value at one time and another value at another time.

B. Exercises in which he tries to estimate the values of objects in another's (the teacher's) hands—the aim of the exercises being to force upon the child the fact that their values are arbitrary, not intrinsic, and that guessing them is a hopeless task.

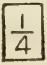
C. Instruction in the systematic way of ascertaining such values.

A. Exercises in Using Objects with Changing Values

"Show me a whole."

"Let us play that it is a *half*."

"Show me a fourth."



"Show me two fourths."

"Show me three fourths."

(The third fourth is obtained by tearing a second paper.)

"Show me a whole again."

"Let us play that it is a fourth."

"Show me an eighth."

(The child exchanges the fourth for two eighths.)

"Show me three eighths."

(He takes another fourth and exchanges.)

"Show me three fourths."

(He exchanges back and then takes a third fourth.)

The above show the form of the work. There should be a dozen or more exercises of this kind—using wholes as halves, thirds, fourths, etc.; a half or third or fourth as a whole; etc.

B. Exercises in Trying to Estimate Values of Objects

The aim of these exercises, as was stated, is to impress the child as strongly as possible, (1) that the number values of objects are assigned arbitrarily by the possessor, and (2) that the unguided guessing at values is wholly profitless.

THE METHOD OF PRESENTATION

These exercises are designed to teach the child that there is one thing that he can *not* do; namely, to guess correctly an arbitrary value fixed by another person and known only to that person. The method therefore must be that of complete, repeated, and merciless baffling of the child in his attempts to gratify a natural desire to *guess right.*

1. *The Objects*

As these exercises have to do with estimating the number values of objects not the pupil's own and not connected with the supply in use by the class, the objects to be used should not be of the same color, if possible, as those that the class has been working with. This change in color makes for fairness because it is, in a way, a warning that the teacher is making use of objects that are not of their A or B or C supply—objects that they would have to give a new letter to, calling them D papers or E papers, if they had them. The difference in color need only be enough to distinguish the objects as plainly *not* from their supply.

2. *Preparation and Arrangement of the Objects*

The objects in this development must, from the very nature of the work, be in the hands of the teacher. In order to provide against delay in going from one step to the next in the work, all of the objects should be prepared beforehand and arranged in a drawer of the desk or table so that they will be available for use in regular order as wanted. The pupils should not know of the preparation of the objects or of their being in the room.

3. Experiences with Arbitrary Values

The aim of this presentation is to defeat the child's every attempt to guess correctly on number values. By this means we shall impress upon him more strongly than by any direct means (explaining) that such efforts are necessarily useless and fruitless.

(*a*) The first object that the teacher takes from the prepared supply in her desk is, perhaps, a paper 3″×6″. It is approximately the size of some of his objects for half.

"What is the number value of this?"

("It is a half.")

"No, it's a *whole*—one."

The teacher now in order to make "one" more emphatic tears the object into two equal parts and shows one of the parts—"This is a half." She then shows the two parts in group form saying, "This is two halves." This serves to emphasize that the original object was considered an integral unit.

(*b*) The second object taken from the drawer may, perhaps, be 1″×3″. The child may see in it a resemblance to one of his regular objects, $\frac{1}{6}$ or $\frac{1}{8}$. He will base his guess on this impression. One pupil may call it $\frac{1}{6}$ another $\frac{1}{8}$ in answer.

"No, this is a half."

"This" (taking another piece from the drawer) "is the other half."

"You seem not to be getting along very well. Let us try again."

(*c*) This time the teacher takes from the drawer a piece about 5″×5″.

"What is this?"

("It is a whole.")

"No, this is a fourth."

"These are the other three fourths"—taking the three pieces from the drawer.

"This is my whole"—taking a 10″×10″ paper from the drawer.

NOTE.—The above show the general plan of the presentation. Each exercise is a defeat for the child. Several additional exercises should be given in the same way, the objects being taken so erratically as to size and assigned value that it is impossible for the child to give the correct estimate. After a few trials the pupil will refuse to guess. This is the evidence that he has learned his lesson—that he recognizes the fact that arbitrary values follow no rule and that there is seemingly no guide to them.

C. How to Answer the Questions of Number Values

The aim here is to teach the child (to *tell* him) how to answer every question of number values of objects in the hands of a person who is not a member of the class and using objects from the common class supply. The child's utter defeat in trying to estimate such values has prepared him to appreciate the fact that only by some definite plan carefully worked out and always followed can one succeed in guessing (?) them.

1. The Presentation

"Children, you have not succeeded in answering a single question as to the number value of objects."

"You go at it wrong."

"Let me tell you how to answer such a question right every time."

"I will take this new object" (taking an object as before—a new one) "and we will see how to answer the question so that it will be right."

"When I ask you what it is, you must not answer but must ask me a question—always the same question."

"The question is, Is it a fraction?"

"I must answer your question."

"If I do not, you must refuse to answer what it is."

The teacher now holds up the object (which, we will assume, is a *whole*), and asks, "What is the number value of this?"

The pupil must be required to reply by asking,

("Is it a fraction?")

"No, it is not a fraction."

The pupil will of course reply,

"It is a whole."

"Yes, it is a whole."

The teacher now takes from the drawer or other place of concealment another object. Let us assume that it is a fourth.

"What is this?"

("Is it a fraction?")

"Yes."

"Now, children, let me tell you what to do and say. It is always the same."

"I answered that it *is* a fraction."

"Now, you must say to me, 'Show me your whole.'"

"Now, I *must* show you my whole or you must refuse to answer my question."

"This is my whole," showing a whole corresponding to the fourth.

"Now, you must come and take my object and my whole and apply the object and others like it, if you wish, until you find what the number value of the object is."

The teacher now has some pupil come forward, take the fraction and the whole, and measure the fraction by applying it to the whole. This done, the pupil states what the fraction is—a fourth.

Another and then another exercise follows, the teacher showing at one time a fraction at another time an integer, the pupils going through the questioning process under her direction.

NOTE.—This is the outline of the method of development of the proper mental attitude in the matter of estimating arbitrary values. The child's sorry experience in trying to guess such values made him recognize the need of a systematic plan of procedure. The teacher must now see to it that the child has numberless opportunities to use this plan accurately. Again and again for weeks this matter of the number values of new objects must be sprung upon the class in an unlooked for connection until the children are fully trained to be on the alert not to put a value on a new object no matter how much it may resemble in shape and size one of the regular objects used in their work.

It is of the greatest importance that the children become thoroughly impressed with this fact of arbitrary values. They are not so impressed—have not learned it—until their impulse to guess at values without thinking the matter out gives way to reason in every case and they approach the value in the way outlined above.

It would be a mistake after teaching this not to spring an exercise under it on the class every few days for weeks and months—until the teacher finds herself unable longer to catch the pupils napping.

2. *Form of the Exercise When Testing the Child*

(*a*) "What is the number value of this?"
("Is it a fraction?")
"Yes."
("Show me the whole.")

The teacher shows a whole. The pupil to whom the question was addressed (or any other child selected by the teacher for answer) comes forward, takes the fraction and the whole, measures the whole with the fraction, and then makes answer—

("It is a third") or ("It is an eighth.")
(*b*) "What is the number value of this?"
("Is it a fraction?")
"No."
("It is a whole.")

SUMMARY.

Exercises in using number objects with one value in one problem and another value in another problem. (*A*.)

This is to establish that the counting value of an object is a matter that the individual may fix or change at will.

Exercises in trying to estimate values of number objects in the hands of another—in the teacher's hands. (*B*.)

The aim in this work is to impress as deeply as possible that the counting value of such objects in constructions is a matter to be *agreed upon* not *guessed at.*

Exercises to teach the child how to proceed systematically to determine the counting values of number objects not his own in advance of any attempt to construct with them or to read constructions by others. (*C* and Note.)

The aim here is to teach the child to refuse to construct or to read constructions with objects not his own until the counting value of the object or objects is ascertained and to teach him how to find out what value has been assigned to them in the problem.

Time required for the work of this chapter, including reviews of Chapters II–XV, five days. Total time for the work of Chapters II–XVI, seventy-one days.

CHAPTER XVII

PARALLEL SERIES—COMMON DENOMINATOR

Topics:—Common. Common Terms. Common Denominator. Lowest Common Denominator. The Terms Sum and Total

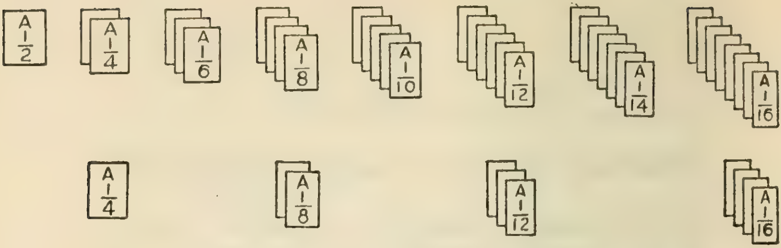
This development is introductory to what the child will learn later to speak of as reducing to a common denominator.

Two fractions as halves and fourths, halves and eighths, thirds and sixths, thirds and fourths, halves and sixths, halves and fifths, fifths and tenths, etc. are dictated for pupils to construct the two series in parallel form.

A. Common Denominator of Two Fractions

1. "Common" Developed

The teacher asks the pupils to construct the "one half" and the "one fourth" series of equivalents on their desks. The child constructs the former series along the back of his desk—as many terms as his desk will hold. He should then be required to construct the latter series in front of it so that the terms of the two series having like pieces (the same name) shall be opposite each other—



This diagram shows the arrangement of the objects on the child's desk—the common units of the two series under each other.

After the construction and after each other construction under this section, the teacher asks the questions—

"What can we make halves into?"

"What can we make fourths into?"

The question suggested in Chapter XV D Note should also be asked.

In the case of these two series, both have fourths, eighths, twelfths, sixteenths, etc. Teach the pupils to call the fourths, eighths, etc., in these series the "common" units.

This word "common"—"common" units—must be applied to all *parallel series* work. "What common units do both series have?" or, in the case of constructions where there are three or four parallel series, "What common units do all have?" "What are the *names* of the *common* units?" "What is the name of the first common unit?" "Of the second?"

NOTE 1.—In this "parallel series," "common," and "common denominator" work, it will be necessary for the teacher in most of her dictations to limit the number of terms to which the series are to be carried. The dictations should specify—"to 24ths," "to 18ths," or "to 35ths," etc.; or, "to the ——th term," meaning that term of the series which has the shortest denominator intervals.

If we say to a child, "Make the full series from $\frac{3}{4}$ and $\frac{5}{8}$ to denominators not higher than 35ths," he should respond with the series to 32ds.

2. *"Common Terms," "Sum" or "Total" of Common Terms*[*]

"You may now make $\frac{1}{2}$ and $\frac{1}{8}$ into parallel series, putting the common units opposite each other."

The constructions made and properly arranged, the teacher asks the questions as above—

"What did you make the half into?"

"What did you make the eighth into?"

"How many common terms might we have to these two fractions?"

(The pupil will discover by *thinking* the extended series that the number is unlimited.)

"What units are common to both series so far as you made them?"

---

[*] Terms having a common denominator, common units, will be spoken of as common terms—common in that their measuring units have the same name.

In answering this last question the pupil should be taught to put his hand on the two groups of eighths saying as he does do, "Eighths are common"; then on the two groups of sixteenths saying as he does so, "Sixteenths are common" then on the two groups of twenty-fourths saying, "Twenty-fourths are common."

"How many eighths in the first series?"

"In the second series?"

"How many in both?"

"How many sixteenths in the first series?"

"In the second?"

"In both?"

"How many 24ths in the first series?"

"In the second?"

"How many in all?"

NOTE 2.—These questions are introduced here not for memory work but as preliminary to addition of fractions later. They are merely to lead the pupil to notice *totals* of common terms. To these questions the teacher should add the request—

"Tell me how many common units of each kind one-half and one-eighth will make."

("They will make five eighths, ten sixteenths, or fifteen twenty-fourths, etc.")

"You may write it on the blackboard."

($\frac{1}{2}$ and $\frac{1}{8}=\frac{5}{8}$, $\frac{10}{16}$, or $\frac{15}{24}$, etc.)

The pupil finds these totals from his objects by putting the objects in each set of terms having like units together just as he has always done in objective adding.

The term *sum* and *total* should be introduced in connection with this work. "The *total* of the first set of like terms is —, of the second —"; or, "The *sum* of the first set of like terms is —, of the second —, etc." The teacher should carefully and methodically interchange these terms—at one time using *sum*, at another time *total*. This will lead the *pupils* to use them interchangeably.

"You may now make $\frac{1}{4}$ and $\frac{1}{8}$ into parallel series to denominators not higher than 34ths, putting the common units opposite each other."

The questions are now asked as before, the child answering the question as to the "Common" groups by putting his hand on them one after the other with, "Eighths are common," "Sixteenths are common," "Twenty-fourths are common," etc.

Ask the same questions as before as to the number of units in the like terms—

"How many 8ths in the series from $\frac{1}{4}$?"

"How many in the series from $\frac{1}{8}$?"

"How many in the total?"

"How many sixteenths in the first series?"

"In the second?"

"The sum of the 16ths?"

"How many 24ths in the series from $\frac{1}{4}$?"

"In the series from $\frac{1}{8}$?"

"In the sum?

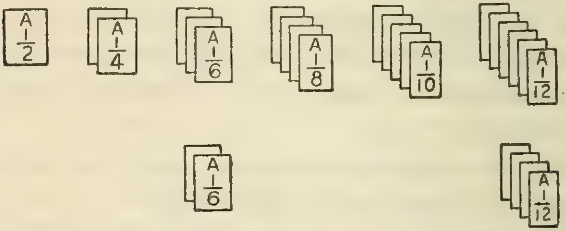
Other constructions follow—$\frac{1}{3}$ and $\frac{1}{6}$, $\frac{2}{3}$ and

$\frac{2}{6}$, $\frac{3}{4}$ and $\frac{1}{8}$, $\frac{2}{3}$ and $\frac{5}{6}$—with the questions and answers as before.

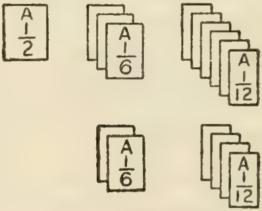
3. Common Denominator

a. Object work—removing terms not having common units.

Constructions of the same kind should be made, the pupils being directed after the constructions to take out and put aside all groups except the leaders and the "common" groups. For illustration, the pupils are told to make the series from $\frac{1}{2}$ and $\frac{2}{6}$ and then to take away the groups whose units are not common. The constructions are—



After the groups have been made as above the pupils take away the fourths, eighths, and tenths, leaving the constructions as follows:



NOTE 3.—"Leader." It will be necessary to teach the pupils to call the basis groups (the $\frac{1}{2}$ and the $\frac{2}{6}$ in the above exercise) the *leaders*—each being the leader of its series.

They should be taught that in removing groups not having common units they must *not* take a leader, even if it is not one of a set of "common" terms.

Three or four exercises of this kind should be given, the child making the full series, arranging the groups of like units opposite each other as above, and then removing the groups not having common denominators excepting, of course, the "leaders." The question suggested in Chapter XV D Note should be used with each full series constructions and also with the series remaining, in each case after taking away the groups whose units are not common, as above.

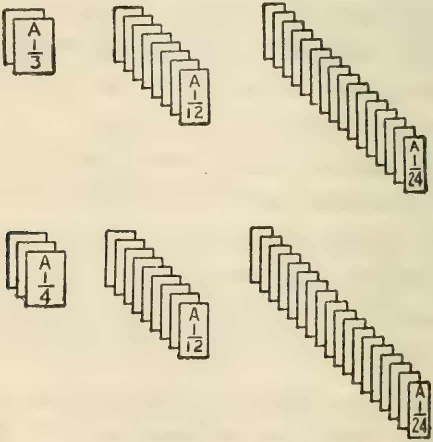
The questions as to the sum of the units in the first common terms, the total of the second common terms, etc. must be asked regularly.

The written statements (See Note 2, page 236) must be required. In this exercise it would be written $\frac{1}{2}+\frac{2}{6}=\frac{5}{6}, \frac{10}{12}, \frac{15}{18}$, etc.

NOTE 4.—The teacher will remember that the pupil finds these totals from his objective work while the construction is before him. He does *not* compute or remember. He counts. Our aim is to accustom him to notice totals (objective) of "common" terms and the *series* of totals.

b. Object work — constructing common terms only.

Exercises are now given in these two-fraction parallel series as above, the pupil to construct only the *common* groups—the groups *not* common to be omitted in the constructions. For illustration, the teacher asks the pupils to "make the parallel series from $\frac{2}{3}$ and $\frac{3}{4}$ leaving out all of the groups that are not common." The constructions will look like those in the exercises just preceding this—those from which the groups *not* common have been removed. The $\frac{2}{3}$ and $\frac{3}{4}$ series will be as follows:



The questions are:—

"What is the series from $\frac{2}{3}$?"

(The pupil in reply gives the *full* series—"$\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$, etc.")

"What is the series from $\frac{3}{4}$ to 30ths?"

(The pupil gives the *full* series.)

"What denominators are common in thirds and fourths to 30ths?"

("Twelfths and twenty-fourths.")

"How many common terms may thirds and fourths have?"

(The child will think this out for himself that the number is unlimited.)

"How many units in all in the first common terms?" "In the second?" "In the third?"

"You may write the sum of the terms in each set of the common terms."

("$\frac{2}{3}+\frac{3}{4}=\frac{17}{12}$, or $\frac{34}{24}$, etc.") (This, of course, is found by the child directly from his constructions.) The question in Chapter XV D Note should be used.

Several similar exercises should be given and the questions asked in each. The question, "What denominators are common?" should take the form also, "What are the common denominators in the series from — and —?" This will bring into use the term "common denominators."

c. Written work without objects—full series.

The pupils should now be given the same work again but without objects. The child simply writes the parallel series placing groups having a common denominator together as in the objective work. For illustration, "You may write the series from $\frac{1}{3}$ and $\frac{1}{4}$ in parallel form to the 9th term"—(the 9th term of the 3ds series).

The child writes—

$\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{5}{15}$, $\frac{6}{18}$, $\frac{7}{21}$, $\frac{8}{24}$, $\frac{9}{27}$, etc.
$\frac{1}{4}$, $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$, $\frac{5}{20}$, $\frac{6}{24}$, $\frac{7}{28}$, etc.

The 12ths and 24ths are *common* and are therefore written opposite each other following the plan of the concrete work. (See pp. 234 and 238.)

"What is the full series from $\frac{1}{3}$ to 30ths?" "From $\frac{1}{4}$ to 30ths?"

"What denominators are common in these series?"

"What can I make both $\frac{1}{3}$ and $\frac{1}{4}$ into?"

This last question puts "common" into a new light. The child has stated many times that he can make thirds into 6ths, 9ths, 12ths, 15ths, etc., and that he can make fourths into 8ths, 12ths, 16ths, etc. He has stated many times what the common denominators are. We now have the question form, "What can I make both into?"

The totals of the "common" terms of the series will be found by the pupil from his written series. These totals of the common terms must be regularly required.

Other exercises of this kind should be given for the child to write without objects. Questions should be asked with each exercise, "What is the full series from —?" "What is the series from —?" "What denominators are common in these series?" or "What are the common denominators in these series?" "What can I make both into?" "What is the sum of the units in the second common term?" "How many common terms may we have in the series from — and —?"

d. Written work without objects—the common terms only.

Exercises should now be given in which the child writes only the terms of the series which have *common* denominators.

"Write four common terms of the series from $\frac{1}{4}$ and $\frac{1}{6}$."

The child writes—

$$\frac{1}{4}, \quad \frac{3}{12}, \frac{6}{24}, \frac{9}{36}, \frac{12}{48}.$$

$$\frac{1}{6}, \frac{2}{12}, \frac{4}{24}, \frac{6}{36}, \frac{8}{48}.$$

"What are the common denominators in fourths and sixths?"

("12ths, 24ths, 36ths, 48ths, etc.")

"What can we make 4ths and 6ths into?"

("12ths, 24ths, 36ths, 48ths, etc.")

"How many common denominators may they have?" or, "How many common terms may the two fractions have?"

The question in Chapter XV D Note also.

"You may make a statement on the board of the totals of the common terms of $\frac{1}{4}$ and $\frac{1}{6}$ to the third common term."

("$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}, \frac{10}{24}$, or $\frac{15}{36}$.")

Very many exercises (20 to 30) should be given before passing to the oral work in 4 *b*,—1st, in writing out full two-fraction series with the common terms together as above; 2d, in writing such series giving only the terms in which the denominators are common; and 3d, in writing such series of the common terms of mixed numbers. The questions and other oral work are important.

4. *Lowest Common Denominator*

a. The terms low, lowest, high, and higher developed.

The introduction of these terms is very simple. The teacher has already without explanation brought into use the expressions first common denominator, second common denominator, third common term, first common term, etc. No explanation was necessary. It was natural to number the common terms in this way.

The pupils are now taught to apply "low" and "high" to denominators. People speak of *half* as being a lower denominator than *third*, of fifth being a higher denominator than fourth or third or half. In $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$, people speak of *half* being the lowest and sixth the highest denominator. These facts are taught the child and he is thus brought to use "low" and "lower" and "lowest" in comparing denominators. "Low" and "high" are now applied in the same way to *common* denominators—the *lowest* common denominator, a *higher* common denominator.

After several exercises in denominators in which they are spoken of as 1st, 2d, 3d, etc., and at the same time compared as to "lowest," "higher," "highest" (of those written), "lower," etc., the child easily connects "lowest" with first. The first denominator of a series is the lowest, the first common denominator is the lowest of the common ones.

"What do we mean by lowest common denominator?"

("We mean the first common denominator.")

"What is the *highest* common denominator?"

The pupil will reason out for himself that there can be no highest.

b. Oral exercises without objects.

Exercises should be given, very many of them, in oral work on common groups and common denominators. It is very important that the dictations and questions should provide the widest possible use of "common denominator" language.

The following questions are to suggest the form and scope of this oral work. The teacher will observe that the exercises (as the questions indicate) relate more particularly to *denominators*, to *common* denominators, and to common denominator *vocabulary*. The power to form complete common series mentally—without a pencil or crayon to record the terms as each is found—will be developed slowly by experience in written work.

"What can we make from halves and fourths?"

("We can make fourths, eighths, twelfths, sixteenths, etc.")

"What terms are common in series from halves and thirds?"

("The common terms are 6ths, 12ths, 18ths, etc.")

"What are the common denominators in the series from fourths and sixths?"

("The common denominators are 12ths, 24ths, 36ths, etc.")

"What is the lowest common denominator in the series from four fifths and five sixths?"

"What are the 2d and 4th common denominators in the series from — and —?"

"What are the numerators of the terms having the lowest common denominator in the series from — and —?"

The question as to totals of the common terms may be used in written but not in *oral* series work.

The intent of this section is to have the child give orally from written or from oral dictations any thing as to denominators that has been developed in the objective and written work in parallel series. The exercises should involve all sorts of two-fraction series up to 10ths, or 12ths—

"I can make 3ds and 4ths into 12ths, 24ths, or 36ths, etc."

"I can make 4ths and 6ths into 12ths, 18ths, 24ths, 30ths, 36ths, 42ds, 48ths, etc."

"I can make 3ds and 5ths into 15ths, 30ths, or 45ths, etc."

"I can make 6ths and 9ths into 18ths, 36ths, or 54ths, etc."

ORAL, WRITTEN, AND OBJECTIVE REVIEWS

Oral and written exercises should be given frequently for weeks in the series work outlined.

Objective work should be given occasionally so that the concrete work will not be lost.

The questions suggested must be in constant use as well as all the terms that have been developed. The teacher must not overlook the question,

"How do you make the series from —?"

("I take — and make each into two equal parts, then I take another — and make each into three equal parts, etc.")

B. Common Denominator of Three or More Fractions

AIM.—This section repeats the work of (*A*) except that it deals with exercises involving more than two fractions—making and writing such combinations of series as halves, fourths, and eighths; halves, fourths, and sixths; thirds, fourths, and sixths; halves, thirds, and fourths; fourths, sixths, and eighths; fifths, sixths, and eighths; thirds, fourths, sixths, and eighths; etc., in parallel form, the terms having common denominators being arranged as before, in columns.

PLAN OF DEVELOPMENT

There are three steps:

(1) Constructions.

(Only a few exercises with objects need be given.)

(2) Written exercises without objects.

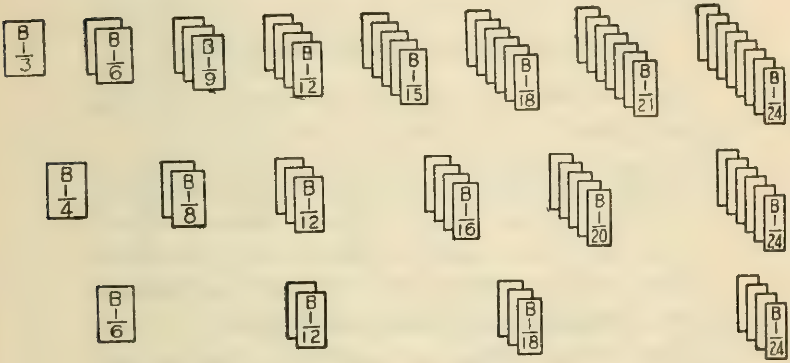
(3) Oral exercises without objects.

1. The Constructions

The order of steps as in *A* is followed here.

a. The full series with the "totals" series of the common terms.

(1) "Make the full series to 25ths from $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ and show the sums of the units in the terms having common names."



After constructing the series as above the pupils write them.

After writing the series the pupil constructs and writes the sums of the terms having common names—the 12ths and the 24ths.

(2) "Make the series of the common terms from $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$, and then the totals of the terms."

(This construction has the three "leaders" and the common terms—the 12ths and 24ths. The teacher will bear in mind, of course, that this being objective work the totals will be written after the construction is completed. The written work after construction is—

The leaders and the two columns of common terms arranged in order, and

The totals of the two columns of common terms.

2. *Written Exercises Without Objects*

(*a*) "Write the full series from $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ to 24ths."

$$\begin{array}{cccccccccccc}
\frac{1}{2}, & \frac{2}{4}, & \frac{3}{6}, & \frac{4}{8}, & \frac{5}{10}, & \frac{6}{12}, & \frac{7}{14}, & \frac{8}{16}, & \frac{9}{18}, & \frac{10}{20}, & \frac{11}{22}, & \frac{12}{24} \\
 & \frac{1}{4} & & \frac{2}{8} & & \frac{3}{12} & & \frac{4}{16} & & \frac{5}{20} & & \frac{6}{24} \\
 & & & \frac{1}{8} & & & & \frac{2}{16} & & & & \frac{3}{24}
\end{array}$$

The only columns of common terms are those having 8ths, 16ths, and 24ths.

The totals series is written—$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$, $\frac{14}{16}$, $\frac{21}{24}$.

(*b*) "Write the common terms of $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ to 24ths, and the sums of the common terms." There are three columns of common terms—

$$\begin{array}{ccccc}
\frac{1}{2} & & \frac{4}{8} & \frac{8}{16} & \frac{12}{24} \\
 & \frac{1}{4} & \frac{2}{8} & \frac{4}{16} & \frac{6}{24} \\
 & & \frac{1}{8} & \frac{2}{16} & \frac{3}{24}
\end{array}$$

The "totals" series is written as in (*a*).

In regular seat work, of which much should be given, the written form should be as follows:

"Write the first three common terms of $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$ with the sum of the lowest."

$$\frac{1}{2} = \frac{6}{12}, \frac{12}{24}, \frac{18}{36}$$
$$\frac{1}{6} = \frac{2}{12}, \frac{4}{24}, \frac{6}{36}$$
$$\frac{1}{12} = \frac{1}{12}, \frac{2}{24}, \frac{3}{36}$$
$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12}, \frac{18}{24}, \frac{27}{36}.$$

The child should have several exercises of this kind to write each day for many days. The question in Chapter XV Note D must not be overlooked.

NOTE.—The questions as to the lowest common denominator, the 2d common denominator, the 3d common denominator, etc., must be asked regularly: also the question of the totals (the sum) of the common terms. All questions suggested in *A* apply here.

3. *Oral Exercises Without Objects*

Oral exercises will be confined to responding to questions like those in *A 4 b*, page 246, with the question on the *numerators* of common terms omitted. The written work in *2 b* above is (because of the fact that the terms are all found mentally the pencil being used merely to record results) in fact oral (mental). The power to change three or more fractions to terms having a common denominator and give their numerators wholly mentally will be acquired only after much experience in work like that in *2 b*.

C. Order and System in the Mental Work in Series Forming

In the nature of things series work is largely mental, the child finding the denominators and the numerators mentally, using the pencil merely to make records of results. There is an orderly way of finding the several common denominators of two or more fractions. It is the aim of this section to train the pupil to do the work in this systematic manner.

1. *The Order of Work*

Write the lowest, the second, and the third common terms of $\frac{1}{2}$, $\frac{1}{6}$, and $\frac{1}{8}$.

In brief, the proper order of thinking in finding the denominators of the common terms is to let the thought center about the highest denominator. In this case it is 8th.

The thinking process is shown in the following questions and answers:

Is 8ths the lowest common denominator? No I can not make 6ths into 8ths.

Is 16ths the lowest common denominator? No I can not make 6ths into 16ths.

Is 24ths the lowest common denominator? Yes, $\frac{1}{2}=\frac{12}{24}$, $\frac{1}{6}=\frac{4}{24}$, $\frac{1}{8}=\frac{3}{24}$.

I must now find the 2d common denominator.

Is 32d this denominator? No, I can not make 6ths into 32ds.

Is 40ths this denominator? No, I can not make 6ths into 40ths.

Is 48ths this denominator? Yes, $\frac{1}{2}=\frac{24}{48}$, $\frac{1}{6}=\frac{8}{48}$, $\frac{1}{8}=\frac{6}{48}$.

A common denominator of any number of fractions is always one of the denominators of the series from the fraction having the highest denominator. By working in this way, always from the highest denominator, the child saves time and works faster.

2. *How to Train the Pupils*

The teacher can train pupils to work with system in this way by devoting a few minutes of each day to blackboard work in which one child at a time goes to the board and talks while he writes and writes while he talks. The child will say just what he writes and write just what he says.

(*a*) Form of oral work.

Let us suppose that the child is sent to the board with the following:

"Change $\frac{1}{4}$, $\frac{2}{5}$, and $\frac{5}{6}$ to their lowest common denominator."

He passes to the board and talks as follows, asking and answering the questions:

"I must find first the lowest common denominator. Is 6ths this denominator? No, I can not make 4ths or 5ths into 6ths. Is 12ths this denominator? No, I can not make 5ths into 12ths. Is 18ths this denominator? No, I can not make 4ths or 5ths into 18ths. Is 24ths the denominator? No, I can not make 5ths into 24ths. Is 30ths the denominator? No, I can not make 4ths into 30ths."

(He goes on in the same way with 36ths, 42ds, 48ths, and 54ths.)

"Can I make them into 60ths? Yes, $\frac{1}{4}=\frac{15}{60}$, $\frac{2}{5}=\frac{24}{60}$, $\frac{5}{6}=\frac{50}{60}$."

(If he had been asked to change them also to the 2d common denominator he would go on with—

"Can I make them into 66ths? No, I can not make 4ths or 5ths into 66ths. Can I make them into 72ds? No, etc.")

The more exercises he writes and talks the more rapid his work. The aim of this talking-writing work at the blackboard is to secure the *habit* of working in this way always from the *highest* denominator.

(*b*) Forms of written dictations.

Change to lowest common denominator and give the totals of the terms:—

$\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{5}$
$\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$
$\frac{5}{7}$ and $\frac{5}{9}$
$\frac{3}{5}$ and $\frac{5}{9}$
$\frac{7}{8}$ and $\frac{7}{10}$
$\frac{1}{3}$, $\frac{4}{5}$, and $\frac{7}{12}$

Change to the lowest and 2d common denominators and give the sum of the units in each set of common terms:—

$\frac{4}{5}$ and $\frac{2}{3}$
$\frac{1}{2}$, $\frac{1}{3}$, and $\frac{3}{4}$
$\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$
$\frac{1}{2}$ and $\frac{4}{7}$
$\frac{2}{3}+\frac{1}{5}$
$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{12}$

Change to 2d and 3d common denominators and find the sum of the units in the 3d common term in each:—

$\frac{1}{2}$ and $\frac{2}{5}$
$\frac{3}{4}$ and $\frac{1}{5}$
$\frac{2}{3}$ and $\frac{1}{7}$
$\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{9}$
$\frac{3}{10}$, $\frac{3}{5}$, and $\frac{1}{3}$
$\frac{5}{10}$ and $\frac{2}{3}$

(*c*) Reverse tests in objective work.

In this exercise the teacher constructs for pupils to read. The aim is to test the child's ability to determine whether the construction is the lowest, 2d, 3d, or 4th "common" term.

(i) The teacher has in mind perhaps, a "common" term for $\frac{1}{3}$ and $\frac{1}{4}$.

"I take this unit—it is $\frac{1}{3}$, and this unit—it is $\frac{1}{4}$."

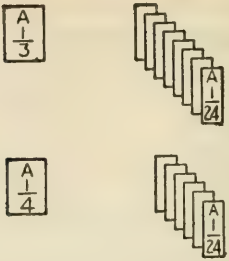
"I take another 3d and exchange it for $\frac{8}{24}$."

"I take another 4th and exchange it for $\frac{6}{24}$."

"What have I done?"

("You have exchanged $\frac{1}{3}$ and $\frac{1}{4}$ for units having their 2d common denominator.")

On page 256 is the form of the teacher's completed construction on her desk—



(ii) The teacher takes in the same way $\frac{1}{5}$ and $\frac{1}{10}$.

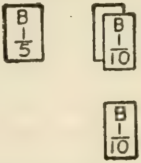
"I take this unit—$\frac{1}{5}$, and this unit—$\frac{1}{10}$."

"I exchange the $\frac{1}{5}$ for $\frac{2}{10}$."

"What have I made?"

("You have made the lowest 'common' terms.")

The following is the teacher's completed construction on her desk—



Several similar exercises should be given, the number being determined by the readiness with which the pupils respond.

(*d*) Reverse tests in written work.

The teacher writes two or more fractions on the blackboard and with them a set of terms "common" to them. The aim is to test the child's ability to determine which "common" term it is.

(i) $\frac{1}{4} = \frac{6}{24}$
$\frac{1}{3} = \frac{8}{24}$
$\frac{1}{6} = \frac{4}{24}$

"What have I done?"

("You have changed the three fractions to their second common denominator")

(ii) $\frac{1}{3} = \frac{20}{60}$
$\frac{1}{5} = \frac{12}{60}$

"What have I done?"

("You have written $\frac{1}{3}$ and $\frac{1}{5}$ in their 4th common denominator."

(iii) $\frac{1}{3} = \frac{3}{9}$
$\frac{1}{9} = \frac{1}{9}$

"What is this?"

("This is $\frac{1}{3}$ and $\frac{1}{9}$ in their lowest common terms.")

(iv) 5ths = 80ths
8ths = 80ths

"What have I written?"

("You have written the 2d common denominator of 5ths and 8ths.")

(*e*) Oral reverse tests.

"To what two fractions would 12ths be common?" *

---

* If a pupil should hesitate on a question of this kind, he should simply be asked to write out the

In answering such a question the child is right if he gives any two of the fractions—halves and 3ds, halves and 12ths, 3ds and 4ths, 6ths and 12ths, etc.

"To what two fractions would 8ths be common?" "9ths?" "4ths?" "6ths?" "15ths?"

"To what three fractions would 8ths be common?" "12ths?" "15ths?" "6ths?" "18ths?" "16ths?"

"If I should change 4ths and 5ths to 40ths what would I have?"

("You would have units in the 2d common denominator.")

3. Exercises in Writing the Changes Horizontally

In all series work thus far the common terms have been arranged in column form. Before the work of Chapter XVIII is taken up the child should have experience in seeing his work detached—written out of the column form, in horizontal form.

(*a*). The introduction.

$\frac{1}{2}$ and $\frac{1}{4} = \frac{2}{4}$ and $\frac{1}{4}$ (Written on the blackboard by the teacher.)

"What have I done?"

---

full equivalent series of the fractions with the "common" terms in columns as before. This work will enable him to answer his question or solve his difficulty without assistance.

("You have written $\frac{1}{2}$ and $\frac{1}{4}$ on their first common denominator.")

$\frac{2}{3}$ and $\frac{3}{4} = \frac{16}{24}$ and $\frac{18}{24}$.

"What did I do?"

("You changed $\frac{2}{3}$ and $\frac{3}{4}$ to units having the 2d common denominator.")

$\frac{1}{6}$ and $\frac{1}{9} = \frac{3}{18}$ and $\frac{2}{18}$.

"What did I write?"

("You wrote $\frac{1}{6}$ and $\frac{1}{9}$ in their lowest common denominator.")

A dozen or more exercises like these should be given—the questions after them varying in form, "What did I do?" "What did I write?" etc.

(*b*.) Written dictations for pupils to complete horizontally,

$\frac{1}{4}$ and $\frac{1}{6} =$

"Write this on your papers and then write them in the 2d common denominator."

("$\frac{1}{4}$ and $\frac{1}{6} = \frac{6}{24}$ and $\frac{4}{24}$.")

Many such exercises should be given:

$\frac{1}{5}$ and $\frac{1}{2} =$ $\frac{3}{4}$ and $\frac{1}{10} =$ $\frac{1}{2}$ and $\frac{4}{9} =$
$\frac{2}{3}$ and $\frac{1}{5} =$ $\frac{1}{2}$ and $\frac{1}{7} =$ $\frac{1}{3}$ and $\frac{1}{8} =$

SUMMARY.

1. The development of "common," "common units," "common terms," "sum of common terms," "total of common terms," "common denominator."

a. Object work—two series. (*A*.)

Exercises in constructions of full series from *two* fractions—arranged parallel, terms having the same denominator in the two series placed opposite each other.

These opposite terms afford the opportunity to introduce common, common units, common terms, parallel series, sum of common terms, total of common terms. (*A*, *1* and *2*.)

Exercises with similar constructions—

(1) Terms not common are taken out afterwards. (*A*, *3 a*.)

(2) Constructions of the common terms only. (*A*, *3 b*.)

Common denominator is applied to the denominators of the terms thus left or constructed. (*A*, *3 b*.)

b. Work without objects—two series.

Exercises in writing out full parallel series from two fractions, the common terms written opposite each other. (*A*, *3 c*.)

Exercises in writing out parallel series—common terms only. The terms common, common terms, etc., as in 2 and 3 are applied.

2. Lowest common denominator is introduced in connection with the parallel series work—the association being with first common denominator, third common denominator, etc. (*A*, *4 a* and *b*.)
   - (*a*) Object work—three or more series. (*B*.)

     This repeats the object work in (*A*) with further application of terms—common denominator, common units, second common denominator, lowest common denominator, etc. (*B*, *1*.)
   - (*b*) Work without objects—three or more series. (*B*, *2* and *3*.)

     This repeats similar work in (*A*) for construction and language experiences.
3. Training in system in the mental work. (*C*.)

   This is to train to find systematically a given common denominator—the 3d, the 5th, the 1st, the lowest, etc.
   - *a*. The training.
     - (1) The order in which the work should be done. (*C*, *1*.)
     - (2) How to conduct the training to this order of work. (*C*, *2*.)

(3) The direct, the reverse objective, the reverse written, and the reverse oral test exercises. (*C*, *2 a* to *e*.)

b. The horizontal order in reductions to common denominator. (*C*, *3*.) This is to provide for the horizontal form of arrangement.

Time required for the work of this chapter, including reviews of Chapters II–XVI, ten days. Total time for the work of Chapters II-XVII, eighty-one days.

CHAPTER XVIII

ADDITION

Topics:—The Need of Common Denominators in Adding. The Arrangement of Written Work in Analyses. Adding Mixed Numbers (Horizontal Form, Vertical Form). The Terms *Add*, *Addition*, and *Sum*

A. Adding Fractional Numbers

Aim.—The work of this section is limited to exercises in which the sum of the fractions to be added is less than a whole because the child has had no experience as yet in changing fractions to larger pieces (lower terms). That subject is to be developed later (Chapter XX). Exercises involving improper fractions and exercises in changing such fractions as $\frac{6}{8}$, $\frac{4}{6}$, $\frac{9}{12}$, etc., to lower terms would, therefore, be out of place at this time.

Addition will require no development steps, no explanations. The child has had all the construction experiences, all the principles, and all the terms needed for the work. In Chapter IV, C he learned what mixed numbers and what impossible mixed numbers are. In Chapter IX, he learned the significance of = and added fractions having a common denominator. Chapters XIII-XVII gave him all necessary experience in changing to common denominator. The terms *add* and *addition*, *sum* and *total* were put into constant use.

1. Objective Work and Solution Forms

a. PREPARATORY—DENOMINATORS COMMON

The teacher should first give (written dictation) three or four exercises in constructions *without* the sign "=," ($\frac{1}{4}+\frac{3}{4}$, $\frac{2}{3}+\frac{1}{3}$, $\frac{3}{8}+\frac{1}{8}$, etc.) and then an equal number in constructions *with* "=." This is for bringing the minds to bear upon constructions and upon the expression "put them together" (=).

NOTE 1.—The manner of conducting this review is of great importance. The construction should follow the writing of the expression by the teacher as quickly as it can be done without disorder. The "putting together" should be completed within a second of the writing of the "=." This form of the work (as in Chapter IX) is as follows:

The teacher writes the expression, as $\frac{1}{2}+\frac{1}{2}$.

The pupils then construct it.

The teacher completes the expression by writing = after it ($\frac{1}{2}+\frac{1}{2}=$).

The pupils respond by putting the objects together.

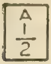
The full expression is then written: $\frac{1}{2}+\frac{1}{2}=\frac{2}{2}$.

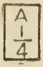
b. SOLUTIONS—DENOMINATORS NOT COMMON

After two or three constructions of exercises having common denominators, the teacher without warning gives the dictation:

(1) $\frac{1}{2}+\frac{1}{4}$.

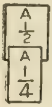
The pupils construct.





The sign = is now added to the dictation ($\frac{1}{2}+\frac{1}{4}=$).

Practically every pupil in the class will quickly put the two objective expressions together.



Each will as quickly recognize this construction as an "impossible" mixed number and will at once separate the objects.

The construction will be for a moment at a standstill. Putting them together is *impossible*, yet = says "put them together."

Note 2.—The teacher will bear in mind that while the necessary changing of $\frac{1}{2}$ and $\frac{1}{4}$ in this exercise is practically what the child did scores of times in Chapters XIII-XVII he will not, in the very nature of things, immediately see the connection. In the sections referred to, the child's thought was directed to the construction of equivalent groups for the sake of *series* or terms having *common denominators*. Here he is trying to *add*. He sees no connection between the two because he has never yet felt the *need* of changing fractions to units having a common denominator in order to add them. All his adding so far has been with fractions already alike in name. New methods, new inventions, new applications of known principles are products of efforts to solve human problems. The plan here is to confront the child with the new situation—that of "putting together" two fractions like $\frac{1}{2}$ and $\frac{1}{4}$ without constructing an "impossible" mixed number. It's a new problem. It is something that will challenge his ingenuity to find a solution. The temptation to help, to explain, will be very strong. The teacher's duty, however, is simply to *wait*, to give the pupil the privilege of *discovering* the "way out." He doesn't need help. He needs quiet and time to think—time to connect, as he surely will, the experiences of Chapters XIII-XVII with the problem before him. Her attitude should be, "Well, you must put the quantities together in some way." She may say as much, but beyond that she must not go.

After two or three minutes of thinking, pupils will be seen one by one beginning to exchange—the indication that the problem has been solved. Some (most pupils) will exchange the $\frac{1}{2}$ for $\frac{2}{4}$ and then will put the groups together. Some will doubtless exchange the $\frac{1}{2}$ for $\frac{4}{8}$ and the $\frac{1}{4}$ for $\frac{2}{8}$ and then combine the groups. *Whatever* the exchange, *the teacher must accept it if it is quantitatively correct.* The time for insisting upon changing to *lowest* common denominator comes later.

"Yes, it is right."

The full written form must now be written on the blackboard by the teacher—the form being that which the child's written seat work later must take:

(i) $\frac{1}{2}+\frac{1}{4}=$
(ii) $\frac{1}{2}=\frac{2}{4}$
(iii) $\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$

(The order of writing this on the blackboard is rather important. It *must* be the order that the child will necessarily use later when he begins to write out solutions. (i) is the original dictation on the blackboard. When the teacher begins to complete the blackboard work after, "Yes, it is right," she writes (ii) then (iii) and then completes (i) by writing in the $\frac{3}{4}$. The completed blackboard work will now be—

$$\begin{aligned} \tfrac{1}{2}+\tfrac{1}{4}&=\tfrac{3}{4} \\ \tfrac{1}{2}&=\tfrac{2}{4} \\ \tfrac{2}{4}+\tfrac{1}{4}&=\tfrac{3}{4} \end{aligned}$$

For those who changed to 8ths the teacher must write the dictation on another part of the board and complete it in the same order as follows:

$$\begin{aligned} \tfrac{1}{2}+\tfrac{1}{4}&=\tfrac{6}{8} \\ \tfrac{1}{2}&=\tfrac{4}{8} \\ \tfrac{1}{4}&=\tfrac{2}{8} \\ \tfrac{4}{8}+\tfrac{2}{8}&=\tfrac{6}{8} \end{aligned}$$

If some members of the class have used 12ths or 16ths, the dictation and the completed statement must be made for them.

The teacher must now ask those who changed the $\frac{1}{2}$ to 4ths, "What did you do?"

(pointing to the blackboard written work). This repeats the work of Chapter XVII, *C*, 3. The child answers, "I gave them their lowest common denominator." The same question is asked of those who used eighths. "I gave them their 2d common denominator." Those who used 12ths would be expected to answer. "I gave them their 3d common denominator."

(2) Another dictation is now given:—

$$\frac{1}{2}+\frac{3}{8}=$$

The pupils construct, each makes exchange as he thinks proper, the groups are "put together," the teacher examines each construction and passes judgment upon it ("impossible" or "your exchange wasn't fair" or "right") and the pupils are asked as before, "What did you do?"

(Since the teacher is accepting any exchange that is not wrong quantitatively, the concrete combinations (the series) are all either right or "impossible." If the child has put the groups together without attempting an exchange, the teacher's comment is "impossible"; if he has exchanged with wrong equivalents, she tells him, "Your exchanges were unfair"; if there is no error, "Right." Errors in constructions will be rare. One can hardly conceive of a child not detecting an "impossible" mixed number.)

The teacher now completes the written work on the blackboard as before:

$$\begin{array}{r}
\frac{1}{2}+\frac{3}{8}=\frac{7}{8}\\
\frac{1}{2}=\frac{4}{8}\\
\frac{4}{8}+\frac{3}{8}=\frac{7}{8}
\end{array}$$

It is understood, of course, that if one or two pupils have used 16ths or 24ths, the full statements should be written—

$$\begin{array}{rcr}
\frac{1}{2}+\frac{3}{8}=\frac{14}{16} & \qquad & \frac{1}{2}+\frac{3}{8}=\frac{21}{24}\\
\frac{1}{2}=\frac{8}{16} & & \frac{1}{2}=\frac{12}{24}\\
\frac{8}{16}+\frac{6}{16}+\frac{14}{16} & & \frac{12}{24}+\frac{9}{24}=\frac{21}{24}
\end{array}$$

The question should be asked in each case here, "What did you do?" The answer should be, "I gave them their 2d common denominator"; or, "I gave them their 3d common denominator."

(3) Other similar exercises should follow but the written work on the blackboard in each case should be done by the pupils. When a pupil is asked to write the full statements on the board he should write it in accordance with his own construction. If, for illustration, in such a dictation as $\frac{1}{4}+\frac{1}{3}$, a pupil changed the units to 24ths he should write the statements for 24ths:

$$\begin{array}{r}
\frac{1}{4}+\frac{1}{3}=\frac{14}{24}\\
\frac{1}{4}=\frac{6}{24}\\
\frac{1}{3}=\frac{8}{24}\\
\frac{6}{24}+\frac{8}{24}=\frac{14}{24}
\end{array}$$

If another pupil changed the units to 12ths his statements would be in 12ths.

NOTE 3.—Oral Work.—When a pupil goes to the blackboard to write these statements, he should be required to "talk" as he writes. The full oral work for $\frac{1}{4}+\frac{1}{3}=$ should be somewhat as follows:

"This is an addition problem. It tells me to add $\frac{1}{4}$ and $\frac{1}{3}$. I cannot put them together because it would make an "impossible" number, it would have more than one kind of fractional units. I must exchange them for equivalent units with a common denominator. Can I make them into fourths? No, I cannot make 3ds into 4ths. Can I make them into 8ths? No, I cannot make 3ds into 8ths. Can I make them into 12ths? Yes. One fourth is equal to three twelfths. I exchange the $\frac{1}{4}$ for $\frac{3}{12}$. One third is equal to four twelfths. I exchange the $\frac{1}{3}$ for $\frac{4}{12}$. Three twelfths and four twelfths are seven twelfths. The sum is $\frac{7}{12}$."

Oral work of this kind should be insisted upon as part of every child's work at the blackboard. The question should be frequently asked, "What do we mean by sum?" Sum means the result in addition.

(4) Exercises for Construction and Blackboard Written Work. These exercises are without suggestion as to which common denominator the child shall use.

$\frac{3}{4}+\frac{1}{8}=$

$\frac{1}{12}+\frac{5}{6}=$

$\frac{3}{4}+\frac{1}{6}=$

$\frac{1}{2}+\frac{2}{5}=$

$\frac{1}{6}+\frac{5}{8}=$

$\frac{1}{7}+\frac{1}{3}=$

(5) Exercises for Construction—Denominator Specified.

Exercises should now be given in which the pupil is directed in each case to use a particular common denominator.

$\frac{2}{3}+\frac{1}{4}=$ (Use the 3d common denominator.)
$\frac{4}{6}+\frac{1}{9}=$ (Use the 2d common denominator.)
$\frac{1}{4}+\frac{1}{6}=$ (Use the 3d common denominator.)
$\frac{1}{5}+\frac{1}{2}=$ (Use the 1st common denominator.)
$\frac{1}{2}+\frac{1}{3}=$ (Use the 4th common denominator.)
$\frac{1}{8}+\frac{1}{6}=$ (Use the 1st common denominator.)

c. Adding with Lowest Common Denominator

The pupils should now be taught that in changing fractions to a common denominator we must always, always use the lowest. The other common denominators are proper for use but people expect us to use the lowest.

Exercises like those in (*b 4*) should now be given, each construction and each blackboard exercise being followed by the question,

"Which denominator did you use?"
("I used the lowest.")
"Why?"
("People expect me to use the lowest.")

$\frac{1}{6}+\frac{3}{4}=$ $\frac{1}{7}+\frac{1}{3}=$ $\frac{2}{3}+\frac{1}{7}=$
$\frac{2}{5}+\frac{1}{6}=$ $\frac{1}{9}+\frac{1}{4}=$ $\frac{1}{4}+\frac{1}{9}=$
$\frac{1}{6}+\frac{2}{9}=$ $\frac{1}{2}+\frac{1}{9}=$ $\frac{1}{7}+\frac{1}{4}=$

2. *Written Work Without Objects*

a. Class Exercises with Oral Work

Exercises should now be given for work without objects. The dictations are written on the blackboard as before, the pupil writes on his paper the complete statements as written on the blackboard heretofore:

$$\begin{aligned} \tfrac{2}{5} + \tfrac{1}{2} &= \tfrac{9}{10} \\ \tfrac{2}{5} &= \tfrac{4}{10} \\ \tfrac{1}{2} &= \tfrac{5}{10} \\ \tfrac{4}{10} + \tfrac{5}{10} &= \tfrac{9}{10} \end{aligned}$$

After completing the exercise in this way a pupil should be sent to the blackboard to write it and "talk" as suggested in Note 3 above.

The pupils should have frequent exercises in going to the blackboard and "talking" one of these exercises. The question should be frequently asked,

"Which denominator did you use?"

("The lowest.")

"Why?"

("People expect us to use the lowest.")

b. Seat Written Work

Many exercises should now be given for written work at their desks:—

$\frac{1}{6} + \frac{5}{2} =$

$\frac{1}{7} + \frac{3}{4} =$

$\frac{2}{5} + \frac{1}{3} =$

$\frac{3}{8} + \frac{5}{12} =$

$\frac{1}{9} + \frac{3}{12} =$

$\frac{1}{8} + \frac{1}{5} =$

$\frac{1}{11} + \frac{1}{9} =$

$\frac{1}{8} + \frac{1}{7} =$

$\frac{1}{4} + \frac{3}{9} =$

B. Adding Mixed Numbers

Aim.—To insure full and clear imagery in mixed number addition by experience in objective adding and afterwards in adding without objects in both *horizontal* and the *vertical* form, giving the full *oral work* in connection with it step by step.

There is nothing new in this excepting the *forms* of the work. When we add $2\frac{1}{4}$ and $3\frac{1}{2}$, we simply put the fractions together (add the fractions), then put the integers together (add the integers), and then put these sums together into an integral or mixed number group.

FORMS OF SOLUTIONS

Adding with mixed numbers occurs in two forms—(i) the *horizontal* ($2\frac{1}{5}+1\frac{1}{2}=3\frac{7}{10}$) and (ii) the vertical

$$\begin{array}{r} 2\frac{1}{5} \\ +1\frac{1}{2} \\ \hline 3\frac{7}{10}. \end{array}$$

Objective work should, therefore, take these two forms.

1. Method of Working

a. LEADER

This work is best done in "class exercises" in which, under the leadership of now one and then another pupil chosen by the teacher to "talk" while he constructs, the constructions are made in as perfect concert as possible, each act of the construction being performed by leader and pupils at exactly the same time under the direction of the leader's oral work, *act* for *word*. Under this plan, if by chance there should be a child in the class who does not fully know how to do some part of the objective work, and turns on that account to see what his neighbors are doing, he is instantly discovered by this being out of concert in his movements. No child can watch and copy a neighbor and move in the construction concert.

b. WRITER

Besides the leader in the concert construction work, another pupil is chosen to write the exercise on the blackboard as the construction proceeds. This is to connect closely the written with the construction work. This pupil's duty is to write the solution in proper form following the leader's *oral* work. If the leader makes a mistake in his "talk"—suggests an exchange for a fraction that is not an equivalent, chooses a wrong common denominator (not the lowest), or makes other mistake—this writer refuses to write the step, thus indicating that he thinks the leader is in error. Writing an error is a count against the writer. When the construction by the leader has been completed, the blackboard will show the written solution in the following form:

$$\text{(i)}\quad \left.\begin{aligned} 2\tfrac{1}{2}+3\tfrac{1}{3}&=5\tfrac{5}{6}\text{*}\\ 2\tfrac{1}{2}&=2\tfrac{3}{6}\\ 3\tfrac{1}{3}&=3\tfrac{2}{6}\\ 2\tfrac{3}{6}+3\tfrac{2}{6}&=5\tfrac{5}{6} \end{aligned}\right\}$$

This should be called the long form of solutions.

Later when the shorter form of solution is adopted, its form would be—

$$\text{(ii)}\quad \left.\begin{aligned} 2\tfrac{1}{2}+3\tfrac{1}{3}&=5\tfrac{5}{6}\\ 2\tfrac{3}{6}+3\tfrac{2}{6}&=5\tfrac{5}{6} \end{aligned}\right\}$$

This will be called the short form. It differs from the long form in that the two intermediate statements are omitted.

c. Detecting Errors

When the "leader" makes a mistake in his oral work, the "writer" refuses to enter it in the written solution on the blackboard. The pupils at their desks who are constructing in concert with the leader's oral work and construction, must in the same manner refuse to make a movement toward construction that is wrong. This breaks the concert. Those who begin to act on a suggestion that is not right show by so acting that they have not noticed the error. Every child that does not notice the mistake is discovered instantly—the others refusing to act.

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* This is copied back from the fourth statement after the latter is completed.

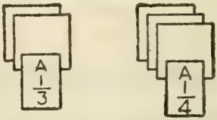
2. *Horizontal Solutions*

a. OBJECTIVE WORK

The teacher gives a dictation—$2\frac{1}{3}+3\frac{1}{4}=$.

She appoints a *leader* and a *writer* at the blackboard. The leader rises in his seat and begins to "talk" and construct as follows, asking and answering the questions as he constructs:

"First, I take $2\frac{1}{3}$, then I take $3\frac{1}{4}$."



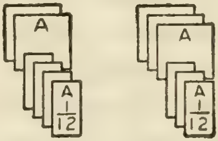
"Can I make the fractions into 4ths? No."

"Can I make them into 8ths? No."

"Can I make them into 12ths? Yes."

"My lowest common denominator is 12ths."

"I take the $\frac{1}{3}$ and exchange it for $\frac{4}{12}$, and the $\frac{1}{4}$ and exchange it for $\frac{3}{12}$."

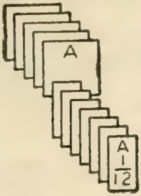


"Four twelfths and three twelfths are seven twelfths."

(He puts the $\frac{4}{12}$ with the $\frac{3}{12}$ and carries the group ($\frac{7}{12}$) to the right on his desk.)

"Two and three are five."

(He puts the 2 with the 3, carries the group (5) to the right, and puts the two groups (sums) together—



"The sum is $5\frac{7}{12}$."

Note.—The teacher will observe that in this horizontal solution the groups to be added are placed horizontally to each other and in adding objectively the groups are carried to the right, just as the written numbers are written horizontally and carried to the right in adding.

The "writer's" board work will be as follows:

$2\frac{1}{3} + 3\frac{1}{4} = 5\frac{7}{12}$ *

$2\frac{1}{3} = 2\frac{4}{12}$

$3\frac{1}{4} = 3\frac{3}{12}$

$2\frac{4}{12} + 3\frac{3}{12} = 5\frac{7}{12}$

(This shows the form of the horizontal objective work. Several exercises of this kind should be given.)

b. Written Work without Objects

It is not necessary to give extended explanations.

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*Written at the conclusion.

(1) Blackboard exercises are given in which one pupil then another goes to the board and does the work of *both* "leader" and "writer." This oral work is the very same oral work that the "leader" gives in the objective work. (See *a* above.) The written work is the very same as that done by the "writer" in the objective work.

(2) Exercises for Seat Work.

Several exercises are written on the blackboard for pupils to *write*—silent work in adding. Some of these exercises should be written out in the long form and others in the short form of solution (*B*, *1 b*, I and II above). Each dictation should specify whether it is to be written out in long or in short form.

3. Vertical Solutions

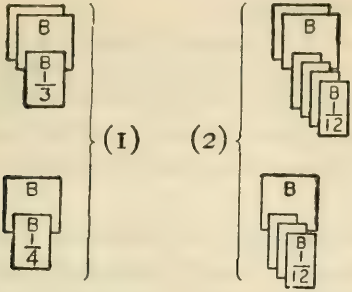
The "vertical" form of solutions follows the plan as to "methods of working" (*B*, *1 a* and *b* and 2 *b*) as that given for the "horizontal" solutions excepting as to the arrangement of the objects. It differs from the "horizontal" in that the objects are arranged like the written expressions—in vertical form.

a. THE CONSTRUCTION AND FORM OF WRITING

(1) Arrangement of the Objects.

$\left.\begin{array}{r} 2\frac{1}{3} \\ +1\frac{1}{4} \\ \hline \end{array}\right\}$ This is a dictation in vertical form.

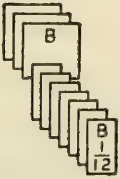
This shows the construction—



The $2\frac{1}{3}$ over the $1\frac{1}{4}$, (1). Opposite these are the two expressions in their lowest common denominator, (2).

The $2\frac{4}{12}$ and the $1\frac{3}{12}$ are "put together" (added) vertically the expression, the "sum"

(3)



taking the form shown in (3).

In changing to lowest common denominator the oral work is like that for the horizontal adding—

"I exchange $\frac{1}{3}$ for $\frac{4}{12}$."
"I exchange $\frac{1}{4}$ for $\frac{3}{12}$."
"$\frac{4}{12}$ and $\frac{3}{12}$ are $\frac{7}{12}$."

"2 and 1 are 3."

"The sum is $3\frac{7}{12}$."

NOTE 2.—The written dictations in the vertical work, the constructions, and the adding, are *vertical.* In all other respects—the "leader," the "writer" at the blackboard, the oral work, and the uses of concrete work—it follows the plan suggested for *horizontal* solutions.

(2) Arrangement of Written Work.

The teacher in developing the objective work should complete the dictation on the blackboard—

$$\begin{array}{rcl} 2\frac{1}{3} & = & 2\frac{4}{12} \\ +1\frac{1}{4} & = & 1\frac{3}{12} \\ \cline{1-1}\cline{3-3} & & 3\frac{7}{12} \end{array}$$

Afterwards, the "writer" at the blackboard in the class concert work arranges the written work in the same form.

b. WRITTEN WORK WITHOUT OBJECTS

(1) Exercises in which one child and then another is sent to the blackboard to do the work of both "leader" and "writer" as in 2 (*b*) above. There should be many such exercises. At least one exercise of this kind should be given every day for several weeks. This is particularly for the oral work.

(2) Seat written work in which many dictations are written on the blackboard and pupils left to write out the solutions on their papers.

4. *Mixed Horizontal and Vertical Exercises—without Objects*

a. WRITTEN DICTATIONS

Some dictations should be given in vertical form:—

$$\begin{array}{r} 3\frac{1}{5} \\ +2\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 2\frac{1}{3} \\ +1\frac{5}{8} \\ \hline \end{array} \qquad \begin{array}{r} 4\frac{2}{7} \\ +3\frac{1}{2} \\ \hline \end{array}$$

When dictations are given in this way the solutions should be in the same form.

Other dictations may be given as follows:

$4\frac{1}{9} + 2\frac{1}{6} =$ $\qquad$ $5\frac{1}{3} + 2\frac{1}{7} =$ (V)
$2\frac{3}{10} + 1\frac{1}{2} =$ (V) $\qquad$ $4\frac{1}{8} + 2\frac{1}{6} =$

This means that when marked (V) the pupil must write it in vertical form and solve it vertically. The pupils should have much of this work.

b. ORAL DICTATIONS

Exercises should be given very frequently in which the teacher dictates the numbers orally, stating in the dictation whether to solve horizontally or vertically.

SUMMARY:

Adding fractional numbers. (*A.*)
  Object work.
    Exercises in constructions with solutions (with =). The aim is to let

    - the pupils discover that to avoid an "impossible" number the fractions must have or be reduced to a common denominator.
    - In this way experience teaches him the addition process. (*A*, *1 b* and Note 2.)
    - The oral work and the written analysis. (*A*, *1 b* and Note 3.)
    - The terms add and addition.
        - Exercises in which the pupil is required to use the *lowest* common denominator. (*A*, *1 c*.)
    - Work without objects.
        - Exercises in adding fractions—using the lowest common denominator. (*A*, *2*.)
- Adding mixed numbers. (*B*.)
    - Object work.
        - Horizontal solutions. (*B*, *2 a*.)
            - The construction form.
            - The oral work.
            - The arrangement of the analysis statements.
        - Vertical solutions. (*B*, *3*.)
            - The construction form. (*B*, *3 a*.)
            - The oral work. (*B*, *3 a* (*1*).)
            - The arrangement of the written analysis. (*B*, *3 a* (*2*).)

Work without objects.
  Horizontal solutions. (*B*, *2 b*.)
  Vertical solutions. (*B*, *3 b*.)

Time required for the work of this chapter, including reviews of Chapters II-XVII, five days. Total time for the work of Chapters II-XVIII, eighty-six days.

CHAPTER XIX

SUBTRACTION

Topics:—The Terms Subtract, Less, Minus, Subtraction, Minuend, Subtrahend, Remainder, Difference, Introduced and Defined. The Need of Common Denominators. Subtraction with Mixed Numbers. "Borrowing"—(1) In Horizontal Form. (2) In Vertical Form

This subject is so simple after the concrete work in common denominator and addition that no development work except for vocabulary will be necessary. The instruction problems are (1) the subdivision of the subject into elementary teaching steps, (2) the oral language of subtraction, and (3) the introduction of terms—subtract, less, minus, remainder, difference, minuend, and subtrahend.

The teaching steps are as follows arranged in their order of sequence:

A. Exercises for subtraction language.

B. Exercises in which the denominator of the minuend only must be changed—$\frac{3}{4} - \frac{1}{8} =$, $\frac{2}{3} - \frac{1}{6} =$, $\frac{4}{5} - \frac{3}{10} =$, $\frac{5}{6} - \frac{5}{12} =$, etc.

C. Exercises in which the denominator of the subtrahend only must be changed — $\frac{5}{6} - \frac{1}{3} =$, $\frac{7}{8} - \frac{3}{4} =$, $\frac{9}{10} - \frac{3}{5} =$, $\frac{3}{4} - \frac{1}{2} =$, $\frac{11}{12} - \frac{2}{3} =$, etc.

D. Exercises in which both denominators must be changed—$\frac{2}{3} - \frac{1}{4} =$, $\frac{3}{4} - \frac{2}{6} =$, $\frac{2}{3} - \frac{1}{5} =$, $\frac{7}{8} - \frac{1}{3} =$, etc.

E. Exercises with mixed numbers in which both denominators must be changed—

(i) Horizontal Solutions

$3\frac{5}{8}-1\frac{1}{3}=$, $4\frac{7}{9}-2\frac{1}{2}=$, $5\frac{7}{10}-2\frac{1}{4}=$.

(ii) Vertical Solutions

$$\begin{array}{rrrrr} 5\frac{6}{7} & 4\frac{2}{3} & 6\frac{5}{8} & 7\frac{2}{3} & 8\frac{5}{9} \\ -2\frac{1}{2} & -1\frac{1}{5} & -1\frac{5}{12} & -2\frac{1}{4} & -2\frac{1}{2}, \text{ etc.} \\ \hline \end{array}$$

F. Exercises which involve "borrowing"—

(i) Horizontal Solutions

$5\frac{1}{2}-2\frac{3}{4}=$, $6\frac{1}{3}-2\frac{3}{4}=$, $9\frac{3}{7}-2\frac{1}{2}$, etc.

(ii) Vertical Solutions

$$\begin{array}{rrrr} 7\frac{1}{4} & 8\frac{3}{8} & 10\frac{4}{3} & 5\frac{1}{7} \\ -2\frac{2}{3} & -1\frac{5}{6} & -3\frac{7}{8} & -2\frac{2}{5}, \text{ etc.} \\ \hline \end{array}$$

G. Exercises without objects.

H. The long way and the short way.

A. Exercises for Subtraction Language.

AIM.—(i) To give the child's thinking a subtracting trend and (ii) to introduce subtract, minuend, subtrahend, subtracted, remainder, and difference.

(1) The teacher writes $\frac{5}{6}-\frac{1}{6}=$ on the blackboard.

One pupil is chosen to stand at his desk and construct and "talk" and one is sent to the blackboard to write—following the plan used in additions.

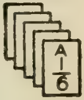
(To the leader) "What does it say?"

("It tells me to take five sixths and take away one sixth.")

"You may construct and talk."

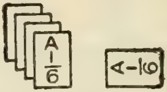
("I take $\frac{5}{6}$.")

He and with him the other pupils take $\frac{5}{6}$ in concert. The "writer" simply points to the $\frac{5}{6}$ that is written on the blackboard—



("I take away $\frac{1}{6}$.")

(Every one takes away the $\frac{1}{6}$ and puts it in proper position. The "writer" points to the written $\frac{1}{6}$.)



(The "writer" completes the written expression $\frac{5}{6}-\frac{1}{6}=\frac{4}{6}$.)

("Five sixths take away one sixth leaves four sixths.")

"Instead of saying take away I want you to use the word *subtract*. Subtract means take away. Instead of the expression *is left* or *leaves* I want you to use the word *remainder* or *difference*. You may use either the one or the other."

NOTE 1.—This is the beginning of the introduction of the subtraction terms. These words as they are introduced one after the other should be written on the blackboard and *should remain there* several days. When this word is given to the class it is written on the blackboard, subtract—take away. As soon as remainder and difference are introduced this written list will be—

Subtract—to take away.

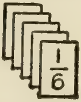
$\left.\begin{array}{l}\text{Remainder}\\ \text{Difference}\end{array}\right\}$ what is left when we subtract

"Show me what you took."

(The pupil puts his original $\frac{5}{6}$ together and replies,

"I took $\frac{5}{6}$."

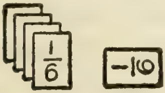
He points to this $\frac{5}{6}$ group as he says it.)



"What did you subtract?"

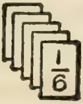
("I subtracted $\frac{1}{6}$.")

He puts the $\frac{1}{6}$ back in its proper place as he says it.



"What did you subtract it from?"

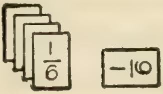
(The child restores the $\frac{5}{6}$ group and replies, pointing to this group,



"I subtracted it from $\frac{5}{6}$.")

"Show me the remainder."

(The child restores the subtraction construction, and, pointing to the proper group ($\frac{4}{6}$) says,



"The remainder is $\frac{4}{6}$.")

"People in speaking of what is subtracted call it the subtrahend."

"Show me the subtrahend."

(The child points to the $\frac{1}{6}$ and says,

"This is the subtrahend.")

When subtrahend is introduced the word is added to the list, "Subtrahend—what we take away." The other terms are added as they are introduced until the list is complete. In the oral work, the oral statements of definitions which are daily required, the pupil gives the meanings as they are written—

"What is meant by the subtrahend?"

("Subtrahend is what we take away when we subtract."

"What does remainder mean?"

("It means what is left when we subtract.")

"Show me the *written* subtrahend."

(The pupil goes to the blackboard and points out the $\frac{1}{6}$ in what the "writer" has written.)

"Show me the subtrahend *construction* on your desk."

"Show me the written remainder."

(The child goes to the blackboard and points to the $\frac{4}{6}$.)

"Show me the remainder construction."

(The child points to his objective $\frac{4}{6}$.)

"Show me the written number that tells what you took."

(The child goes to the blackboard and points to the written $\frac{5}{6}$.)

"Show me with your objects what you took."

(The $\frac{5}{6}$ construction is restored.)

"People call the number that we *subtract from*, the number we *take*, the *minuend*."

("Minuend—the number that we subtract from," is now put into the list of terms on the blackboard.)

(2) $\frac{7}{8}-\frac{2}{8}=$ (Written on the blackboard.)

The teacher selects a "leader" and a "writer."

"You may begin." (This to the leader.)

NOTE 2.—This means that the leader is to begin at the beginning of the oral work and lead the class through the full construction—the writer at the same time to point to the terms of the dictation, as the leader proceeds, and to complete the full written statement as the leader performs the act of subtraction.

The leader begins to talk—

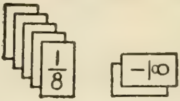
("This tells me to take $\frac{7}{8}$ and subtract $\frac{2}{8}$.")

("Take $\frac{7}{8}$.")

All take $\frac{7}{8}$ with him and the "writer" points to the $\frac{7}{8}$ of the dictation.

("Subtract $\frac{2}{8}$.")

All take away $\frac{2}{8}$ and the "writer" writes the $\frac{5}{8}$, completing the dictation $\frac{7}{8} - \frac{2}{8} = \frac{5}{8}$.

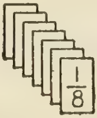


"Show me the subtrahend."

(All point to the $\frac{2}{8}$ of the construction, the "writer" to the written $\frac{2}{8}$.)

"Show me the minuend."

All restore the objects to the original form "taken" and point to it, the "writer" pointing to the $\frac{7}{8}$ of the dictation.



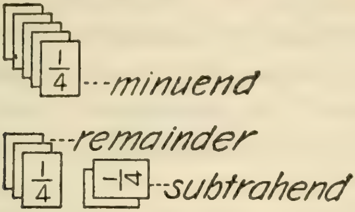
"Show me the remainder."

All make again the subtraction construction and point to the $\frac{5}{8}$, the "writer" simply pointing to the $\frac{5}{8}$ of the full written statement.

(3) This is to be a class exercise, no "leader" or "writer."

"I am thinking of an expression in subtraction. The minuend is $\frac{5}{4}$, the subtrahend is $\frac{2}{4}$. Construct it."

(If a pupil hesitates, the only suggestion to him should be, "Think what *minuend* is, and what *subtrahend* is.")



"Each of you may now write the full expression." ($\frac{5}{4}-\frac{2}{4}=\frac{3}{4}$.)

"What is the $\frac{3}{4}$?"

("It is the remainder.")

"What is the $\frac{2}{4}$?"

("The subtrahend.")

"What is the $\frac{5}{4}$?"

("The minuend.")

(4) "I am thinking of a subtraction expression. The difference is $\frac{5}{7}$, the subtrahend $\frac{3}{7}$. Construct it."

"What is the minuend?"

("It is $\frac{8}{7}$.")

"You may write the full expression."

($\frac{8}{7}-\frac{3}{7}=\frac{5}{7}$.)

Several exercises like (3) and (4) should be given to establish the terms already introduced and to introduce *minus* and *less*. The list of terms on the blackboard will now include—

$$\left.\begin{array}{l}\text{Subtract}\\ \text{Minus}\\ \text{Less}\end{array}\right\}\text{take away.}$$

The dictations may be in the forms—

"I am thinking of $\frac{8}{3}$ minus $\frac{2}{3}$. Construct it."

"I am thinking of $\frac{5}{9}$ less $\frac{3}{9}$. Construct it."

"Construct an expression that will say, 'Minus $\frac{2}{5}$,' tell me what it is, and then write it."

(For this last, of course, there would be various constructions—$\frac{8}{5}-\frac{2}{5}=$, $\frac{6}{5}-\frac{2}{5}=$, $\frac{10}{5}-\frac{2}{5}=$, etc.)

Exercises on *written dictations without* constructions should be given also.

$\frac{9}{12}-\frac{4}{12}=$

"What is this?" (pointing to the $\frac{9}{12}$.)

("It is the minuend.")

"What is this?" (pointing to the $\frac{4}{12}$.)

("The subtrahend.")

B. Exercises in which the Denominator of the Minuend only must be Changed

(1) The Need of a Change Discovered.

(*a*) $\frac{3}{4}-\frac{1}{8}=$

(This is written on the blackboard.)

The teacher asks one of the pupils to tell what it says.

("It tells me to take $\frac{3}{4}$ and subtract $\frac{1}{8}$.")

"What is the subtrahend?"

"What do we mean by subtrahend?"

"What is the minuend?"

"Meaning of minuend?"

"You may construct."

(Each pupil will take $\frac{3}{4}$ and attempt to construct.)

The plan of construction is to leave the child to work out for himself the problem of this subtraction. It is very plain that the child will find himself unable to subtract (to take) $\frac{1}{8}$ from objects that are 4ths. Most children will exchange one of the 4ths for $\frac{2}{8}$ and attempt to make the construction in that way. As soon, however, as he tries to construct he finds the concrete "remainder" to be a group having two fourths and one eighth —an *impossible* mixed number. This will show him that his subtraction is wrong.

The teacher must resist every temptation to suggest or assist. The child can and should work out the problem. It may take three, or four, or five minutes; but he will discover for himself that he must exchange all of the three fourths for eighths and that then he can subtract and have a *possible* difference. The problem has in it but one unknown element—that of how many of his fourths to change into eighths. The child should have the privilege and pleasure of discovering this.

As soon as the pupils have completed the construction they should be asked to state what they did.

("I exchanged the $\frac{3}{4}$ for $\frac{6}{8}$ and then subtracted.")

The blackboard dictation should now be completed—

$$\frac{3}{4} - \frac{1}{8} = \frac{5}{8}*$$
$$\frac{3}{4} = \frac{6}{8}$$
$$\frac{6}{8} - \frac{1}{8} = \frac{5}{8}$$

This form of work follows that which the pupil used in addition.

(2) The Oral Work Developed.

(*a*) $\frac{5}{3} - \frac{1}{6} =$

The pupil states what the dictation tells him; the teacher questions as to minuend, subtrahend, etc.; and the pupil then proceeds with the construction.

There will be less delay with this construction. It will take but a minute for the child to decide that he must change all of his 3ds to 6ths and then subtract. The construction finished, a pupil is sent to the blackboard to complete the written statement in accordance with the construction—

$$\frac{5}{3} - \frac{1}{6} = \frac{9}{6}$$
$$\frac{5}{3} = \frac{10}{6}$$
$$\frac{10}{6} - \frac{1}{6} = \frac{9}{6}$$

---

*Copied back from the 3rd statement.

"How did the change of $\frac{5}{3}$ to $\frac{10}{6}$ affect the two fractions in the expression as to their denominators?"

("It gave them their lowest common denominator.")

(*b*) $\frac{3}{2}-\frac{3}{4}=$

The same plan as in (*a*) is to be followed with this, pupils stating what it says the teacher questioning as to names of the terms, the pupils constructing, and the written statement completed—

$$\frac{3}{2}-\frac{3}{4}=\frac{3}{4}$$
$$\frac{3}{2}=\frac{6}{4}$$
$$\frac{6}{4}-\frac{3}{4}=\frac{3}{4}$$

"What did the change of $\frac{3}{2}$ to $\frac{6}{4}$ do?"

("It gave the fractions their lowest common denominator.")

(*c*) $\frac{3}{5}-\frac{5}{10}=$

In this expression the teacher should use a "leader" and a "writer." The "leader" should give the full oral work, constructing as he talks, the others constructing with him. The "writer" should point out and write the terms as the construction goes on.

("This tells me to take $\frac{3}{5}$ and subtract $\frac{5}{10}$. I take $\frac{3}{5}$. I cannot subtract 10ths from 5ths so I exchange the $\frac{3}{5}$ for $\frac{6}{10}$. The minuend and subtrahend now have a common denominator. $\frac{6}{10}$ minus $\frac{5}{10}$ leaves $\frac{1}{10}$. My remainder is $\frac{1}{10}$.")

The "writer" following the "leader's" oral work has completed the written work—

$$\frac{3}{5} - \frac{5}{10} = \frac{1}{10}$$
$$\frac{3}{5} = \frac{6}{10}$$
$$\frac{6}{10} - \frac{5}{10} = \frac{1}{10}$$

Several more exercise, should be given for construction, oral work, and writing—the teacher being careful to require full oral work with each exercise, including the fullest possible use of the subtraction terms that have been learned.

C. Exercises in which the Denominator of the Subtrahend only must be Changed

$\frac{9}{10} - \frac{3}{5} =$

This work should be taken up directly by a "leader" and "writer." It is very rare that a pupil hesitates on this step.

("This tells me to take $\frac{9}{10}$ and subtract $\frac{3}{5}$ from it. I take $\frac{9}{10}$. I cannot take 5ths from 10ths so I think the $\frac{3}{5}$ to $\frac{6}{10}$. The fractions now have their lowest common denominator $\frac{9}{10}$ less $\frac{6}{10}$ leaves $\frac{3}{10}$. My remainder is $\frac{3}{10}$.")

NOTE.—It will be necessary to help the pupil over this part of the language. He has no $\frac{3}{5}$ to change. It is not constructed except as it is in the minuend before the act of subtraction. The child *thinks* the subtrahend *always*. So here, he is taught to say, "I think the $\frac{3}{5}$ to $\frac{6}{10}$."

The "writer" has followed the construction as follows:

$$\begin{aligned} \tfrac{9}{10} - \tfrac{3}{5} &= \tfrac{3}{10} \\ \tfrac{3}{5} &= \tfrac{6}{10} \\ \tfrac{9}{10} - \tfrac{6}{10} &= \tfrac{3}{10} \end{aligned}$$

Several additional exercises should be given until the construction and oral work are established satisfactorily.

D. Exercises in which Both Denominators must be Changed

$\frac{2}{3} - \frac{1}{4} =$

This step is a repetition in a single exercise of *B* and *C*. It involves nothing new except a fourth equation in the *written* statement. The teacher writes the dictation and appoints a "leader" and a "writer."

("This tells us to take $\frac{2}{3}$ and subtract $\frac{1}{4}$. Take $\frac{2}{3}$. I cannot take 4ths from 3ds. I must change them to a common denominator—the lowest. Can I change them to 8ths? No. Can I make them into 12ths? Yes. I exchange $\frac{2}{3}$ for $\frac{8}{12}$. I *think* $\frac{1}{4}$ to $\frac{3}{12}$. $\frac{8}{12}$ minus $\frac{3}{12}$ leaves $\frac{5}{12}$. The difference is $\frac{5}{12}$.")

When the leader says, "I exchange $\frac{2}{3}$ for $\frac{8}{12}$," the "writer" writes the $\frac{2}{3} = \frac{8}{12}$. When he says, "I think $\frac{1}{4}$ to $\frac{3}{12}$," the "writer" writes the $\frac{1}{4} = \frac{3}{12}$. With the "$\frac{8}{12}$ minus $\frac{3}{12}$ leaves $\frac{5}{12}$," the $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$ is written. The blackboard written work is now—

$$\begin{array}{rcl}
\frac{2}{3}-\frac{1}{4} & = & \\
\frac{2}{3} & = & \frac{8}{12} \\
\frac{1}{4} & = & \frac{3}{12} \\
\frac{8}{12}-\frac{3}{12} & = & \frac{5}{12}
\end{array}$$

Several such exercises should follow, care being given to the oral and the written work. The questions involving the subtraction terms —minuend, subtrahend, minus, remainder, etc.—should be asked in connection with each exercise.

E. Exercises with Mixed Numbers in which Both Fractions must be Changed

These exercises are like those under *D* so far as concerns fractions. No outlines, therefore, need be given.

$3\frac{5}{8}-1\frac{1}{3}=$

The "leader" and "writer" plan as before.

("This tells me to take $3\frac{5}{8}$ and subtract $1\frac{1}{3}$. Take $3\frac{5}{8}$. I cannot take 3ds from 8ths and so I change the fractions to their lowest common denominator. Can I make them into 16ths? No. Can I change them to 24ths? Yes. I exchange $\frac{5}{8}$ for $\frac{15}{24}$. I now have $3\frac{15}{24}$. I *think* the $\frac{1}{3}$ to $\frac{8}{24}$. My subtrahend is now $\frac{8}{24}$. $\frac{15}{24}$ less $\frac{8}{24}$ leaves $\frac{7}{24}$. 3 less 1 leaves 2. My remainder is $2\frac{7}{24}$."

The written work by the "writer" following the oral work is —

$3\frac{5}{8} - 1\frac{1}{3} = 2\frac{7}{24}$ (This remainder is written after the solution is completed.)

$3\frac{5}{8} = 3\frac{15}{24}$

$1\frac{1}{3} = 1\frac{8}{24}$

$3\frac{15}{24} - 1\frac{8}{24} = 2\frac{7}{24}$

Many exercises under *D* and *E* should be given. The questions should be asked with each exercise. There should be also a full use of subtraction terms.

F. Exercises which Involve "Borrowing"

It must be assumed that the pupil long before this time was taught subtraction with integral numbers in which he used the principle of "borrowing"—

$$\begin{array}{r} 34 \\ -16 \\ \hline \end{array} \qquad \begin{array}{r} 92 \\ -33 \\ \hline \end{array} \qquad \begin{array}{r} 182 \\ -\ 49 \\ \hline \end{array} \qquad \begin{array}{r} 512 \\ -134 \\ \hline \end{array}$$

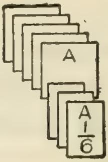
Borrowing, therefore, with fractions will not seem to him to be strange. He will recognize it in the following exercises as a new application of an old principle.

1. Horizontal Subtraction

$5\frac{1}{2} - 2\frac{2}{3} =$

The "leader" and "writer" are selected as before.

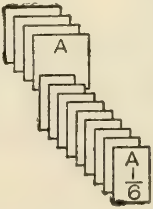
("This tells us to take $5\frac{1}{2}$ and subtract $2\frac{2}{3}$. I take $5\frac{1}{2}$. I cannot take 3ds from halves; therefore I change them to a common denominator. Can I change them to 6ths? Yes. I exchange $\frac{1}{2}$ for $\frac{3}{6}$. I now have $5\frac{3}{6}$. I think $\frac{2}{3}$ to $\frac{4}{6}$. My subtrahend is now $2\frac{4}{6}$. $\frac{4}{6}$ from $\frac{3}{6}$ I cannot take; so I take one of my wholes, leaving 4 wholes, and exchange it for $\frac{6}{6}$.")

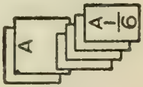


(As he says this he takes up one of the 5 wholes, puts it back into the supply, and brings back to his minuend group $\frac{6}{6}$.)

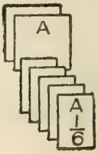
("This makes my minuend $4\frac{9}{6}$. $\frac{9}{6}$ less $\frac{4}{6}$ leaves $\frac{5}{6}$. 4 less 2 leaves 2.")

The construction is now—





("My remainder is $2\frac{5}{6}$.")



The "writer's" work at the blackboard will show the following at the moment the "leader" finishes his oral work as above—

(i) $5\frac{1}{2}-2\frac{2}{3}=$
(ii) $5\frac{1}{2}=5\frac{3}{6}$
(iii) $2\frac{2}{3}=2\frac{4}{6}$
(iv) $5\frac{3}{6}-2\frac{4}{6}=$
(v) $4\frac{9}{6}-2\frac{4}{6}=2\frac{5}{6}$

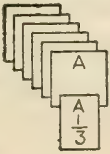
The "writer" now completes the written work by inserting the $2\frac{5}{6}$ as the second member of equation I and IV.

2. *Vertical Subtraction*

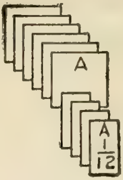
The suggestions for horizontal and vertical additions in Chapter XVIII *B*, 2 *a* and *b*, are sufficient to guide the teachers in these forms of solutions in subtraction. The oral work in vertical subtraction is, of course, exactly like that for the horizontal.

$$\begin{array}{r}6\frac{1}{3}\\-2\frac{3}{4}\\\hline\end{array}$$

The construction has but one group and, of course, can be neither horizontal nor vertical—

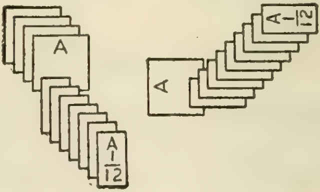


The construction after changing to lowest common denominator is—



The construction after "borrowing" is simply the group $5\frac{16}{12}$.

The construction after subtraction is the regular form for such expression.



(The teacher will note that on account of the fact that subtraction, unlike addition and multiplication, shows practically only one term, the minuend, the terms horizontal and vertical refer to the forms of the *written* work and not to the form of the *concrete* expression.)

The "writer's" work on the blackboard will appear as follows when the leader reaches the conclusion of his oral work:

$$\begin{array}{rcrcr} 6\frac{1}{3} & = & 6\frac{4}{12} & = & 5\frac{16}{12} \\ -1\frac{3}{4} & = & 1\frac{9}{12} & = & 1\frac{9}{12} \\ \hline & & & & 4\frac{7}{12} \end{array}$$

G. Exercises without Objects

These are of two kinds:

(1) Seat work in written solutions, horizontal and vertical, in the *full form* suggested for the "writer's" work in *F*, *1* and *2* above. The teacher gives a dictation on the blackboard of a dozen or more examples, each pupil writing the solutions on his paper as a study exercise. The dictation should be in form as follows:—

$6\frac{1}{4}-2\frac{2}{3}=$ (h)
$5\frac{1}{6}-1\frac{1}{2}=$ (v)
$5\frac{1}{2}$ less $2\frac{3}{5}=$ (v)
$7\frac{1}{5}$ minus $3\frac{2}{3}=$ (v)
$7\frac{3}{4}$ less $2\frac{5}{8}=$ (h)
$8\frac{3}{7}-4\frac{1}{5}=$ (h)
$9\frac{5}{6}$ take away $2\frac{1}{3}=$ (v)
$7\frac{1}{8}$ from $9\frac{1}{6}=$ (h)
From $8\frac{1}{9}$ take $2\frac{1}{3}=$ (h)
$9\frac{1}{8}$ minus $2\frac{5}{12}=$ (v)

The minuend is $7\frac{1}{4}$, the subtrahend $2\frac{5}{6}$.

The (h) and (v) mean horizontal form and vertical form respectively. If an example has no suggestion after it as to (h) and (v), the child exercises his choice as to the form of solution.

(2) Exercises for oral work.

This oral work is exactly like that of the "leader" in *F*, *1* and *2*. The fact that he does not "take" objects does not affect the form of his oral work. An example $8\frac{5}{8}-2\frac{11}{12}=$ (h), perhaps, is written on the blackboard or dictated orally, and a pupil is sent to the board to "talk" and "write." He does the work of "leader" and "writer," the written work going along with the oral. One pupil, then another, and then another are sent to the board for an exercise of this kind. The oral directions should make use of less, minus, minuend, and subtrahend, as suggested for the written dictations in (1) above.

H. "Long Way" and "Short Way"

All the solutions thus far in subtraction have been in what we may call the "long" form—

$$7\tfrac{5}{12}-2\tfrac{5}{8}=(4\tfrac{19}{24}) \qquad \text{(i)}$$
$$7\tfrac{5}{12}=6\tfrac{34}{24} \qquad \text{(ii)}$$
$$2\tfrac{5}{8}=2\tfrac{15}{24} \qquad \text{(iii)}$$
$$6\tfrac{34}{24}-2\tfrac{15}{24}=4\tfrac{19}{24} \qquad \text{(iv)}$$

This "long way" in horizontal solutions should be used exclusively for two or three weeks. At the end of that time the pupils should be required to solve one then another exercise by leaving out of the written work statements (ii) and (iii)—making the changes to the common denominator directly from (i) to (iv). The four-statement solution—(i), (ii), (iii), and (iv)—may be called the "long way"; the two-statement solution, the "short way."

In the vertical solutions the "short way" has the first and third statements (P. 302). Hereafter the dictations will be (h.) (l. w.), or (v) (l. w.), or (v.) (s. w.), or (h.) (s. w.), meaning horizontal long way or vertical long way, etc.

Later, the "short way" will consist of a single line—the child changing to a common denominator and subtracting mentally and writing only the difference. The "long way" will then have *two* statements only. The *oral* work at the blackboard when the pupil "talks" and "writes" will not change. Under all circumstances it remains as in the objective work —in unabridged form.

All pupils in a class cannot begin the single line solutions at one time. When the teacher wishes to try this form she will simply ask the pupils to complete the first statement without writing the second. One or two pupils will do it. Later, others will write it as easily. It will come gradually. Some pupils will require more "long way" experience than others before they are able to meet the mental demands of the single line work. In the meantime, such pupils must not by word, look, or act be discredited. Slowness is far from being a sign of weakness.

In the end, all simple solutions should be made in this short form.

SUMMARY:

1. Object work.
   a. Exercises with fractions having a common denominator to introduce the subtraction terms—subtract,

minuend, subtrahend, subtraction, remainder, difference, minus, less; to define these terms; and to apply them to the *objective* expression as well as to the *written*. (*A*.)

b. Exercises in which *one* of the denominators must be changed—that of the minuend.

In these exercises the child finds a need of a common denominator to avoid "impossible" numbers. He makes the change in order to effect the subtraction. (*B*.)

He is taught to arrange the solutions in analysis form. (*B*.)

c. Exercises in which the denominator of the subtrahend must be changed.

The child finds that to avoid an "impossible" number a change must be made as before. (*C*.)

d. Exercises in which both fractions must be changed in reducing to a common denominator. (*D*.)

e. Exercises with mixed numbers—both denominators changing in the common denominator. (*E*.)

f. Exercises involving "borrowing"—

1st. Horizontal form. (*F*, *1*.)
2nd. Vertical form. (*F*, *2*.)

2. Work without objects.
   - *a.* Exercises in solutions, using fractions and mixed numbers. (*G.*)
     - (1) Horizontal solutions—fractions, mixed numbers.
     - (2) Vertical solutions—mixed numbers.
   - *b.* Exercises in "short way" and in "long way" solutions. (*H.*)

Time required for the work of this chapter, including reviews of Chapters II-XVIII, five days. Total time for the work of Chapters II-XIX, ninety-one days.

CHAPTER XX

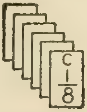
REDUCING FRACTIONS TO LOWER DENOMINATORS

Topics:—The Form of Objective Reductions. The Terms Reduce, Series of Lower Equivalents, Series of Higher Equivalents, Higher Terms, Lower Terms, Lowest Terms, Proper Fraction, Improper Fraction. Orderly Methods of Reductions

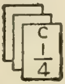
In the presentation of this work the word "reduce" should be "exchange." We exchange $\frac{2}{2}$ for 1 *whole*, $\frac{6}{8}$ for 3 *fourths* ($\frac{3}{4}$), $\frac{10}{6}$ for 1 *whole* and 2 *thirds* ($1\frac{2}{3}$), etc. The objective work is that of *exchanging*, the word "reduce" having no meaning expressive of construction acts except as it takes the place of "exchange" later as a synonym.

1. Imagery

The form of constructions in reducing to lower terms or lowest terms, if the imagery is to be accurate, is of utmost importance. In exchanging, for illustration, $\frac{6}{8}$ for larger (lower) fractional units the child must exchange a group ($\frac{2}{8}$) for a unit ($\frac{1}{4}$), then another group ($\frac{2}{8}$) for a unit ($\frac{1}{4}$), then another group for a unit, and then he must make one group of the three 4ths. The group to be exchanged was $\frac{6}{8}$.—

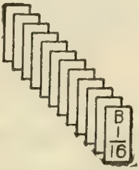


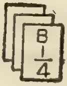
The form of the objects after exchange is—

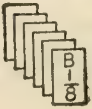


the group $\frac{6}{8}$ becoming by the exchange $\frac{3}{4}$.

In exchanging, for further illustration, $\frac{12}{16}$ for larger (lower) units, the exchange may be by 4's—$\frac{4}{16}$ for $\frac{1}{4}$, another $\frac{4}{16}$ for $\frac{1}{4}$, and the other $\frac{4}{16}$ for $\frac{1}{4}$, making a new group of three 4ths (the original group, $\frac{12}{16}$—

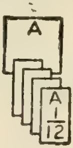


the new group, $\frac{3}{4}$— ); or by 2's—
$\frac{2}{16}$ for $\frac{1}{8}$, another $\frac{2}{16}$ for $\frac{1}{8}$, another $\frac{2}{16}$ for $\frac{1}{8}$, etc., making the new group of $\frac{6}{8}$—

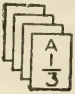


In either case the exchange gives *lower* units, the former giving *lowest* units.

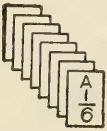
In exchanging $\frac{16}{12}$ for larger (lower) units, the exchange may be by 12's, the new group being $1\frac{4}{12}$—



or by 4's, the new group being $\frac{4}{3}$—



or by 2's, the new group being $\frac{8}{6}$—



A second exchange in the $1\frac{4}{12}$ ($\frac{4}{12}$ for $\frac{1}{3}$) would give a final group of $1\frac{1}{3}$.

A second exchange in the $\frac{4}{3}$ ($\frac{3}{3}$ for 1) would also give a final group of $1\frac{1}{3}$.

A regrouping of $\frac{8}{6}$ by 6's and then by 2's, or by 2's and then by 3's, would give the same form, $1\frac{1}{3}$.

The essential feature of the construction is that the original group is made into *one equivalent group*.

2. *Method of Presentation*

The subject and the presentation are simple and will require but a comparatively short period of time. It must be given, however, in three distinct steps:

a. Reduction of Fractions that are Equivalent to Integers—$\frac{8}{8}$, $\frac{12}{6}$, $\frac{7}{7}$, $\frac{27}{9}$, $\frac{10}{10}$, etc.

In the reduction of such fractions, there is no tendency—hardly an opportunity—to make more than one form of grouping in a construction. In exchanging $\frac{12}{12}$, for illustration, the child will group the whole $\frac{12}{12}$ by 12's, by 6's, by 4's, by 3's, or by 2's. It does not occur to him to regroup a *part* of the $\frac{12}{12}$ in one way (by 2's perhaps) and the rest by 4's or 3's.

Construction with such fractions, because of their simplicity and directness, is properly selected, therefore, for the introductory work —the development of the mechanical features of concrete reduction (how to exchange), orderly methods of exchange, the proper arrangement of equivalent terms, and the *oral* work which must go with reductions to lower and lowest units.

b. Exchanging Groups that are Equivalent to Mixed Numbers.

This class of reductions is made a separate step because in many of such exercises there is a new element in construction—a part of the group to be reduced is exchanged in one form, the rest being regrouped differently or not at all. The first (lowest) term, for illustration, of the series of lower equivalents of $\frac{10}{8}$ is $1\frac{1}{4}$, the construction of which requires the exchange of part of it ($\frac{8}{8}$) by 8's, and the rest ($\frac{2}{8}$) by 2's. In the same way, in reducing $\frac{9}{8}$, part of it ($\frac{8}{8}$) is exchanged by 8's and the rest ($\frac{1}{8}$) is not exchanged but used in its original form—$\frac{9}{8} = 1\frac{1}{8}$.

A new element, therefore, enters into the reduction of fractions having mixed number equivalents, making necessary, from a teaching standpoint, a separate development step.

c. Exchanging Groups which are Less than a Whole.

Proper fraction reductions—$\frac{6}{8}$, $\frac{15}{18}$, $\frac{15}{20}$, $\frac{10}{12}$, etc.—must be given a separate development step in order that, after his work with fractions equivalent to mixed numbers, the child's construction experiences will force upon his attention that the two-form and the part-group forms of exchanges, which are necessary there, are impossible in fractions of this kind.

A. Exchanging Groups Equivalent to Integers

1. *Introduction*

"Take $\frac{8}{8}$."

(Each child takes $\frac{8}{8}$ putting the objects, of course, in proper group form.)

"I want each of you to exchange your objects for the equivalent in larger pieces."

Some will doubtless exchange for 1; others, possibly, for $\frac{2}{2}$; others for $\frac{4}{4}$. The teacher must examine the construction as quickly as possible to see (1) that the exchanges are for *equivalents* and (2) that the result of the exchange in each case is a single group. If the child has exchanged for something that is not an equivalent the proper suggestion is, That was not a fair exchange. The child should find and correct his errors. If the result has not been arranged as one group the information must be given the child that he must so arrange it.

"The $\frac{8}{8}$ is one number. What you exchange them for must also be one number."

The teacher now passes upon the constructions,—now one, now another:

To one pupil—"You exchanged your $\frac{8}{8}$ by 4's—$\frac{4}{8}$ for $\frac{1}{2}$, the other $\frac{4}{8}$ for another $\frac{1}{2}$. You have $\frac{2}{2}$. Which is greater, the $\frac{8}{8}$ or the $\frac{2}{2}$?"

("They are equivalents.")

The teacher asks the pupil to write on the blackboard a statement that the $\frac{8}{8}$ and the number thus exchanged for it are equivalents —($\frac{8}{8}=\frac{2}{2}$).

To another pupil—"You exchanged your $\frac{8}{8}$ by 8's—$\frac{8}{8}$ for 1. Which is greater, the $\frac{8}{8}$ or the 1?" "You may write on the blackboard that the $\frac{8}{8}$ and what you have are equivalents." ($\frac{8}{8} = 1$).

To another pupil—You exchanged your $\frac{8}{8}$ by 2's—$\frac{2}{8}$ for $\frac{1}{4}$, another $\frac{2}{8}$ for another $\frac{1}{4}$, another $\frac{2}{8}$ for another $\frac{1}{4}$, another $\frac{2}{8}$ for another $\frac{1}{4}$. Which is greater in quantity, the $\frac{8}{8}$ or the $\frac{4}{4}$?

"You may write on the blackboard that the $\frac{8}{8}$ and what you have are equivalents ($\frac{8}{8} = \frac{4}{4}$)."

"Take $\frac{12}{12}$ and exchange for larger pieces."

Then follows the same examination of individual constructions as to exchange for equivalents and arrangement in single groups, and the same form of comment as to form of exchange by 12's, by 6's, etc. This time, as before, one child and then another should be sent to the blackboard to write in full the result of his exchange—$\frac{12}{12} = \frac{4}{4}$, $\frac{12}{12} = 1$, or $\frac{12}{12} = \frac{3}{3}$, etc. The question, How many groups do we have after the exchange? should now be developed and asked repeatedly in constructions. The child's reply to it is, of course, that he has one group. Several other similar exercises should follow—$\frac{10}{10}$, $\frac{6}{6}$, $\frac{12}{12}$, $\frac{16}{8}$, $\frac{20}{10}$, $\frac{21}{7}$, etc.

The forms of dictations should be—

$\frac{9}{9}$ to larger units.
$\frac{12}{6}$ to lower units.
$\frac{12}{12}$ to lower denominator.

(The terms denominator, lower units, etc., are more or less familiar to the child from his work on fraction series. He will understand the meaning. They should have constant use here. The term *higher units* must also be used. It helps to give meaning to *lower units*.

In connection with this work there are the following very important

2. *Teaching Points*

a. Exchanging by Units.

In the child's construction work he must be required to exchange by single groups. In other words, when he exchanges $\frac{8}{8}$ by 2's, he must exchange $\frac{2}{8}$ for $\frac{1}{4}$, then another $\frac{2}{8}$ for $\frac{1}{4}$, etc. In exchanging by 4's he exchanges $\frac{4}{8}$ for $\frac{1}{2}$ then the other $\frac{4}{8}$ for $\frac{1}{2}$.

b. Making as Many Exchanges as Possible.

When the child has made one exchange (in $\frac{12}{12}$, exchanging by 6's perhaps), he must be taught to look for other possible forms of exchange, and, if he finds one, to take another group ($\frac{12}{12}$ in this case) and make the exchange. The child must find all the ways in which, in a given group, exchanges may be made, and must make all of such exchanges.

c. Arranging Exchanges in Order—Lowest First.

In the exchange from $\frac{20}{10}$ the child's objects must be arranged in order—2, $\frac{4}{2}$, $\frac{10}{5}$. If not made in such order the results must afterwards be so arranged. These results arranged in order form a *series* of equivalents.

d. Oral Work of Exchanging.

There should be oral work and written work with "leader" and "writer" just as in other number developments. The "leader" stands at his desk and constructs and talks. The "writer" at the blackboard does the writing as the "talking" progresses.

The oral work for $\frac{6}{6}$ is:

"I take $\frac{6}{6}$. I exchange by 6's—$\frac{6}{6}$ for 1. I take another $\frac{6}{6}$. I exchange by 3's—$\frac{3}{6}$ for $\frac{1}{2}$ and the other $\frac{3}{6}$ for $\frac{1}{2}$. I have $\frac{2}{2}$. I take another $\frac{6}{6}$. I exchange by 2's—$\frac{2}{6}$ for $\frac{1}{3}$, another $\frac{2}{6}$ for $\frac{1}{3}$, the other $\frac{2}{6}$ for $\frac{1}{3}$. I have $\frac{3}{3}$. The series of lower equivalents has three terms—1, $\frac{2}{2}$, and $\frac{3}{3}$. The lowest is 1."

When the "leader" says at the beginning, "I take $\frac{6}{6}$," the "writer" writes $\frac{6}{6}$. As he says, "$\frac{6}{6}$ for 1" the "writer" writes 1, making the blackboard show $\frac{6}{6} = 1$. When the leader says "I have $\frac{2}{2}$," the writer adds this to the written work ($\frac{6}{6} = 1 = \frac{2}{2}$). When the leader says, "I have $\frac{3}{3}$," the writer adds this to the written work ($\frac{6}{6} = 1 = \frac{2}{2} = \frac{3}{3}$).

e. Making Exchanges in Order—Lowest First.

The child must discover in advance the possible exchanges in a group in order (lowest equivalent term first) and must make his exchanges in that order. In $\frac{15}{15}$, his constructions must be in the order 1, $\frac{3}{3}$, $\frac{5}{5}$, etc. The results of exchanges must be arranged invariably in this order—from the lowest up. Many such exercises must be given.

Note.—Teacher and pupils must use the terms "series" and "terms"—series of lower equivalents, two terms in the series of lower equivalents, the lowest term in the series of lower equivalents, etc. The child must be led to recognize that he now has two kinds of series of equivalents. He must learn to speak of them as *higher* equivalents and *lower* equivalents. This involves the teaching of "*higher*" and "*lower*" as applied to fractions. Halves are lower than 3ds, 6ths are higher than 3ds or 4ths or 5ths, 10ths are lower than 12ths, 10ths are higher than 8ths or halves or 9ths, etc. The term "reduce" must be gradually introduced to take the place of *change*. The dictation form, *Change* $\frac{24}{18}$ to its series of lower equivalents, becomes, *Reduce* $\frac{24}{18}$ to its series, etc. The teacher should make frequent use of the term *reduction* in referring to this work so as to put it into the child's vocabulary as quickly as possible. *Reduce* and *reduction* are of frequent use in number, hence their importance in the child's ready vocabulary. *Reduce* means *change*—change in the sense that it has been used in this connection.

f. Forms of Questions to be Used.

These questions illustrate the kind to be used with all constructions here and later with written work after objects have been laid aside.

(i) In how many ways may 8ths be exchanged? (8ths may be exchanged by 8's, giving wholes; by 4's, giving halves; and by 2's, giving 4ths.)

(ii) When we exchange 10ths by 2's, what do we get? By 5's? By 10's?

(iii) What are the possible lower terms from 12ths? (Wholes, halves, 3ds, 4ths, and 6ths.)

(iv) What is the series of lower equivalents from —— (the group to be reduced)?

B. Exchanging Groups Having Mixed Number Equivalents

1. The Development Plan

The teacher writes a fraction as $\frac{10}{8}$ on the blackboard and directs the pupils in terms of *reduce*—

"Reduce to lower terms."

The pupils will exchange first by 8's because this is what they have been doing. The construction on almost every desk will be that of $1\frac{1}{4}$—exchanging $\frac{8}{8}$ for 1 and the $\frac{2}{8}$ for $\frac{1}{4}$. Some however, will have $1\frac{2}{8}$.

"Take another $\frac{10}{8}$ and construct another term of a series if you can."

The constructions will now be, doubtless, $1\frac{2}{8}$ and $1\frac{1}{4}$—those having $1\frac{1}{4}$ for the first construction now having $1\frac{2}{8}$, and those with $1\frac{2}{8}$ at first now having $1\frac{1}{4}$. Each pupil's two constructions must now be arranged in proper order—the "lowest first." This is to be done by the pupil himself—the teacher simply calling attention to the fact that there is a proper order, that they have been taught what that order is, and that that order must always be observed.

"How did you" (to one pupil and then another) "make your first exchange?"

("By 8's, exchanging $\frac{8}{8}$ for 1, and by 2's, exchanging $\frac{2}{8}$ for $\frac{1}{4}$. This gave me $1\frac{1}{4}$.")

"Your second exchange?"

("By 8's, exchanging $\frac{8}{8}$ for 1. This made my number $1\frac{2}{8}$.")

Another pupil with the other order of construction will answer these two questions in the reverse order.

"Which of these two terms is the lower?"

("The $1\frac{1}{4}$.")

"Who of you made $1\frac{1}{4}$ for your first construction?"

Another dictation as $\frac{8}{6}$ should now be given and the constructions made in the same manner. Another dictation is given as $\frac{9}{8}$. (In this construction the pupils will be unable to find a second term of the series.) Another dictation as $\frac{16}{12}$ is given. The constructions here should show $1\frac{1}{3}$, $1\frac{2}{6}$, $\frac{4}{3}$, and $\frac{8}{6}$ arranged in that order.

2. *The Oral Work*

It is very important here, as in fact in all work, that the oral work with "leader" and "writer" be made a constant feature of the development. A leader's oral work should be—

"I take $\frac{8}{6}$. I group by 8's, exchanging $\frac{8}{8}$ for 1. I group the $\frac{2}{8}$ by 2's, exchanging the $\frac{2}{8}$ for $\frac{1}{4}$. My number is $1\frac{1}{4}$."

"I take another $\frac{8}{6}$. I group by 8's, exchanging $\frac{8}{8}$ for 1. I leave the $\frac{2}{8}$ unchanged. My number is $1\frac{2}{8}$."

"I take $\frac{9}{8}$. I group by 8's, exchanging $\frac{8}{8}$ for 1. $\frac{1}{8}$ is in its lowest form. My number is $1\frac{1}{8}$. This is the lowest form."

The "writer's" blackboard written work should be $\frac{8}{6} = 1\frac{1}{4}$, $1\frac{2}{8}$; $\frac{9}{8} = 1\frac{1}{8}$; or $\frac{16}{12} = 1\frac{1}{3}$, $1\frac{2}{6}$, $\frac{4}{3}$, $\frac{8}{6}$, following corresponding oral work by the leader.

3. *Questions to be Used*

The questions suggested in *A*, *2 f* above must not be overlooked here. Other forms of questions:

How many terms in the series of lower equivalents of ——?

What is the lowest term in the series of lower equivalents from ——? What is the 2d term in the series of lower equivalents from ——?

4. Exercises

Many exercises in reductions of improper fractions, as above, should be given.

5. Improper Fractions and Proper Fractions—the Terms Introduced

Teach pupils to speak of these fractions as "improper." Teach to define "improper" as applied to fractions: "An improper fraction is one that would have a whole number or mixed number equivalent." "A proper fraction is one that is less than a whole."

C. Exchanges with Proper Fractions

There is nothing new in development here. The feature of this step is that the child is left to learn through construction experiences that a lower term by *two* forms of grouping, as in fractions whose equivalents are mixed numbers, is impossible—that two-form groupings produce impossible numbers.

1. Object Work

The teacher dictates a proper fraction as $\frac{12}{16}$ on the blackboard.

"You may construct the first term of the series of lower equivalents."

(Some may try exchanging by 8's and the remainder by 4's. This would be natural after the experiences of the last step. This attempt, however, to put the resulting $\frac{1}{2}$ and $\frac{1}{4}$ into *one group*—to make *one number* as all lower term constructions require—will show him that the exchanges are impossible. The $\frac{1}{2}$ and $\frac{1}{4}$ form an impossible number.)

Everyone's final lowest term construction will be $\frac{3}{4}$—an exchange by 4's. In the same way, taking another $\frac{12}{16}$ for a second term of the series, the pupil will find but one form of exchange possible—an exchange by 2's, making $\frac{6}{8}$.

Another dictation is written as $\frac{10}{18}$.

In the same way the child finds that the lowest equivalent term (and the only lower term) is $\frac{5}{9}$—an exchange by 2's.

Other dictations are given and the constructions made in the same manner. Some of these dictations should be fractions that are already in their lowest terms—the pupil to discover and state the fact.

The oral work with "leader" and "writer" as suggested in *A*, *2 d;* and in *B*, *2* must be a feature of the work also.

The questions suggested in *A*, *2 f*, and in *B*, *3* above must be in constant use.

The definitions of proper fractions and improper fractions must be called for frequently.

2. *Exercises without Objects*

Very many exercises without objects in reductions to lower terms should be given. Some of these dictations should require simply the *lowest* equivalent term, others should require the *full series* of lower terms. All the questions suggested for use with the objective work must be continued here just the same as if objects were in use. Oral work like that with objects—work in which the "leader" goes to the blackboard to do the work of both "leader" and "writer" but without the objects—must be required very frequently, first one child and then another being sent to the blackboard with an exercise. These exercises should involve reductions of improper fractions and proper fractions—all of the forms of work presented in *B* and *C* above.

D. Applications of Reductions—Seat Exercises in Addition

This subject of addition was developed in Chapter XVIII, but in that work we were careful to use no exercises where the sum had an improper fraction. Now that changing to lowest terms has been developed, we are ready to complete the work of addition as an application of reduction work. Development work is unnecessary because there is no new imagery or language. Class exercises with objects should be given, however, occasionally for months for the sake of keeping the construction work fresh. Except for this purpose objects should not be used. The work of this section will differ from that in Chapter XVIII only in reducing each *sum* to its lowest terms. In the addition of mixed numbers, exercises must be given, as in Chapter XVIII, in "vertical" and in "horizontal" solutions. Pupils should be able to use either form as directed.

FORMS OF EXERCISES

There are three forms of exercises under this section—

(1) Class exercises with objects for the construction work and for the oral language. Exercises with "leader" and "writer" as in Chapter XVIII *B*, *1 a* and *b*. Such exercises are to be given frequently.

(2) Class exercises *without objects* for oral work—one child and then another going to the blackboard to do the work of both "leader" and "writer."

$3\frac{3}{4}+2\frac{5}{6}=$

("I take $3\frac{3}{4}$ and $2\frac{5}{6}$. I cannot add $\frac{3}{4}$ and $\frac{5}{6}$ because the pieces are not alike—they do not have a common denominator. I exchange $\frac{3}{4}$ for $\frac{9}{12}$ and $\frac{5}{6}$ for $\frac{10}{12}$. $\frac{9}{12}$ and $\frac{10}{12}$ are $\frac{19}{12}$. This is equivalent to $1\frac{7}{12}$. I write down the $\frac{7}{12}$. 3 and 2 and 1 are 6. I have $6\frac{7}{12}$.")

The horizontal solutions should be written out in "short form" (Chap. XVIII, *B*, *1 b* (II)) occasionally as the teacher shall direct.

(3) Seat work without objects in adding mixed numbers. This work should involve horizontal and vertical solutions. The dictations should indicate whether the solution is to be written out horizontally or vertically. In the dictations for horizontal solutions the teacher should indicate that it should be in "long form" if she wishes that form used. If no form is suggested in such dictations the pupil should use the "short form." (Chap. XVIII *B*, *1 b* (II).)

Forms of dictations:

$2\frac{1}{2}+3\frac{5}{6}=$

$5\frac{1}{4}+1\frac{2}{3}=$ (long way).

$4\frac{4}{5}+2\frac{1}{3}=$

$$\begin{array}{r}6\frac{1}{2}\\+2\frac{4}{7}\\\hline\end{array}\qquad\begin{array}{r}8\frac{1}{6}\\+2\frac{7}{8}\\\hline\end{array}\qquad\begin{array}{r}5\frac{3}{5}\\+2\frac{7}{8}\\\hline\end{array}$$

SUMMARY:

1. Object Work.
   - *a.* An outline of the form that object work must necessarily take in reductions to lower terms. (*1.*)
   - *b.* Constructions in reducing to lower terms fractions whose lowest form is an integer—fractions like $\frac{8}{8}$, $\frac{16}{4}$, etc. The aims of this work are—

(i) That the pupil may learn to work out systematically the problem of constructing every possible lower equivalent of any fraction of this kind. (*A*, 2 *a* and *b*)

(ii) That he may learn to arrange these equivalents in order, the lowest first. (2 *A*, 2 *c* and *e*.)

(iii) To put into the working vocabulary the terms reduce, series of lower equivalents, series of higher equivalents, higher terms, lower terms, and lowest terms. (*A*, Note.)

c. Constructions in the reduction of fractions whose lowest form is a mixed number. (*B*.)

This has for its aims—

(i) The further use of the terms.

(ii) The special form of the reduction with such fractions.

d. The terms proper fraction and improper fraction developed and defined. (*B*, 5.)

The reduction of proper fractions to lower and to lowest forms. (*C*.)

2. Work without Objects. (*C*, 2)

a. The application of reductions to lower and to lowest terms to addition of fractions using problems in which the sums or amounts are improper fractions or proper fractions not in lowest forms. (*D.*) (See Chapter XVIII, Aim.)

b. Adding mixed numbers—horizontal and vertical forms. (*D*, (*3*).)

Time required for the work of this chapter, including reviews of Chapters II–XIX, ten days. Total time for the work of Chapters I–XX, 101 days.

CHAPTER XXI

MULTIPLICATION WITH INTEGRAL MULTIPLIERS

Topic:—The Forms and the Language of the Object Work

INTRODUCTORY NOTE.—Multiplication of fractions from the standpoint of objective development is divided into two parts—(1) exercises in which the multiplier is an integer and (2) those in which it is a mixed number or fraction. The former in nature and consequently in form of concrete expression is so much like addition that it logically follows that subject as a sequel or supplement. In addition to this, partition development requires a knowledge of this kind of multiplication. Multiplication when both factors are fractions or mixed numbers must be made for a similar reason a supplement to partition.

This is the reason for the introduction of this part of multiplication here following addition and preceding partition. There is little in the section that is essentially new. It practically adds to the work of Chapter XVIII only vertical solutions and the reduction of mixed number products to lowest terms. No introductory development work will be necessary.

A. Objective Work

This is of two kinds:

(i) Reading statements forwards and backwards.

(ii) Class exercises with "leader" and "writer" for oral work and the forms of written solutions—horizontal and vertical.

1. *Reading Statements Forwards and Backwards*

The teacher writes statements on the blackboard and calls upon individuals to read them forwards and backwards.

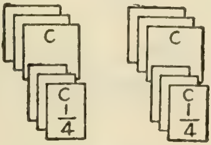
$2 \times 3\frac{3}{4} =$

("This means *two* three and three-fourths.")

("It means *two times* three and three-fourths.")

("It means three and three-fourths *two times*," or "twice".)

"How would you construct $2 \times 3\frac{3}{4}$?"



("I would make $3\frac{3}{4}$ twice.")

"Each of you may make it."

2. *Class Exercises for Oral and Written Work—Horizontal Form*

$3 \times \frac{1}{2} =$

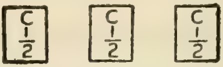
("This means three times $\frac{1}{2}$.")

("It means $\frac{1}{2}$ three times.")

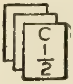
("The = means that after the groups are made we must put them together.")

"You" (to some pupil) "may do what it tells you, and talk."

("First I make three halves.



I put them together.")



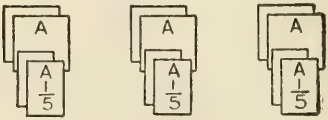
(In work of this kind, a "leader" constructing and talking, there must be a "writer" at the blackboard.)

$3 \times 2\frac{2}{5} =$

"You" (designating some pupil) "may construct and talk."

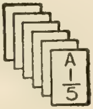
("This says 3 times $2\frac{2}{5}$. It says three two and two-fifths. It says that $2\frac{2}{5}$ is to be made three times.")

("I take $2\frac{2}{5}$ three times.")

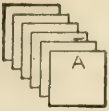


("I put them together.")

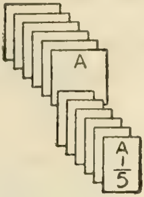
("Three times $\frac{2}{5}$ are $\frac{6}{5}$.")



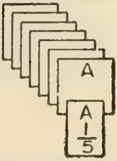
("Three times 2 are 6.")



("I have 6 and $\frac{6}{5}$.")



("I exchange $\frac{5}{5}$ for 1. I now have 7 and $\frac{1}{5}$.")



The writer's work at the blackboard accompanying this construction and oral work will be as follows:

$3 \times 2\frac{2}{5} =$

(This is written when he first goes to the board. It is simply the problem that is to be constructed. Nothing is added to this when the "leader" says, "I take $2\frac{2}{5}$ three times," nor when he says, "put them together." When he says, "Three times $\frac{2}{5}$ are $\frac{6}{5}$," the writer's board work is given the form $3 \times \frac{2}{5} = \frac{6}{5}$. When he says, "Three times 2 are 6," the writer's board work reads $3 \times 2\frac{2}{5} = 6\frac{6}{5}$. When he says, "I now have 7 and $\frac{1}{5}$," the writer's board work is made to read—$3\times 2\frac{2}{5}=6\frac{6}{5}=7\frac{1}{5}$.)

(The above shows the general form of the objective, the oral, and the solution forms of the work of this part of this section.)

3. *Class Exercises for Oral and Written Work—Vertical Form*

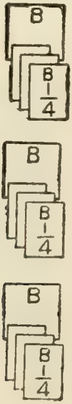
In the vertical solutions the oral work of the "leader" will be exactly the same as for the horizontal. The construction and the "writer's" work, however, will be vertical.

a. THE "LEADER'S" WORK

$$\begin{array}{r}1\frac{3}{4}\\ \times 3\\ \hline\end{array}$$

("This means three one and three fourths.")
("It means $1\frac{3}{4}$ taken 3 times.")
("I take $1\frac{3}{4}$ three times.")

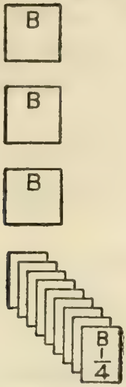
(1)



This shows the seat construction.

("I put them together.")
("Three times $\frac{3}{4}$ are $\frac{9}{4}$.")

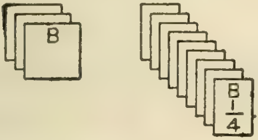
(2)



This shows how (1) is changed when the pupil says, "three times $\frac{3}{4}$ are $\frac{9}{4}$."

("Three times 1 are 3.")

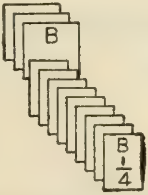
(3)



This shows the construction with "Three times 1 are 3"; (2) is changed to (3).

("I have $3\frac{9}{4}$.")

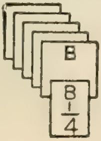
(4)



This shows the change from (3) as the pupil says, "I have $3\frac{9}{4}$."

("I exchange $\frac{8}{4}$ for 2. I have $5\frac{1}{4}$.")

(5)



The final form.

(The above shows the oral work, the construction, and the direction and form of the "putting together." The teacher will note that the objective work is like the written solutions in direction—both vertical.)

b. The "Writer's" Work

The "writer's" blackboard work should be as follows:

$1\frac{3}{4}$ $\times 3$	The problem copied before the "leader" begins.
$1\frac{3}{4}$ $\times 3$ $\frac{9}{4}$	As the blackboard will look when the leader says, "Three times $\frac{3}{4}$ are $\frac{9}{4}$."
$1\frac{3}{4}$ $\times 3$ $3\frac{9}{4}$	As the blackboard will be made to look as the "leader" says, "I have 3 and $\frac{9}{4}$."
$1\frac{3}{4}$ $\times 3$ $3\frac{9}{4} = 5\frac{1}{4}$	As the blackboard is made to look as the "leader" says, "I have $5\frac{1}{4}$."

B. Seat Work without Objects

Exercises in which one pupil and then another goes to the blackboard to do the work of "leader" and "writer." Such exercises should be given frequently for many weeks. These exercises should be about equally divided between the horizontal and the vertical forms. The oral work should be exactly the same as if objects were being used. The child reads the problem forwards and then backwards and then begins, "I take —— —— times, etc."

The written work follows the form used by the "writer" in the class work with objects.

KINDS OF EXERCISES

The dictations should be as follows:

$$\begin{array}{lrr}
3 \times \frac{1}{2} = & 3\frac{3}{4} & 2\frac{5}{8} \\
4 \times \frac{2}{3} = & \underline{\times 4} & \underline{\times 5} \\
7 \times 1\frac{3}{8} = & & \\
5 \times 2\frac{2}{3} = & 3\frac{5}{6} & 5\frac{7}{9} \\
6 \times \frac{3}{5} = & \underline{\times 5} & \underline{\times 4}
\end{array}$$

SUMMARY:

1. Introductory note stating the relation of the subject to addition and partition.
2. Object work. (*A*.)
   - *a*. Exercises in oral statements of the meanings of written expression—reading them forwards and backwards—with constructions.
   - *b*. Horizontal solutions—objective, oral, and written work. (*A*, *2*.)
   - *c*. Vertical solutions—objective, oral, and written work. (*A*, *3*.)

3. Work without objects. (*B*).
    - Written work with analysis.
        - Exercises for horizontal solutions.
        - Exercises for vertical solutions.

Time required for the work of this chapter, including reviews of Chapters II–XX, four days. Total time for Chapters II–XXI, 105 days.

CHAPTER XXII

EVEN PARTITION WITH ANALYSIS

Topic:—The Construction and the Analysis forms.

A. Preliminary Considerations

1. *What the Child Knows*

As fraction development work (or any other line of development for that matter) advances, it is assumed that the teacher keeps each step fresh by occasional review exercises. On this assumption we may safely take up the work of partition where Chapter X left it. We may assume that the child is fresh in the construction forms and in the language of *even* partition with integral and fractional numbers. He knows what any given even partition expression, as $\frac{3}{4}$ of 12, means; will construct it; and will give the oral work ("I take 12, make them into 4 equal groups, show three of these groups, and hide one"). He knows also what "=" means.

This knowledge covers these particulars:

(1) How many objects to take.

(2) How many groups to make.

(3) How many groups to show.

(4) How many groups to hide.

(5) The groups must be *equal.*

(6) The "=" tells us to put the "seen groups" together.

2. *Questions on Partition*

There are several questions that the teacher must ask again and again during this construction work whether with or without analysis. The questions will force deeper into the mind the essential features of partition:

What tells how many to take?

What tells how many groups to make?

What tells how many groups to show?

How many groups must we hide?

What does —— (the number) tell?

What does —— (the denominator) tell?

What does —— (the numerator) tell?

When we say, "Take ——, make it into —— equal groups, show ——, and hide ——," what is the word that we must emphasize most strongly? ("Equal.")

3. *The Partition of Chapter X as Modified by Chapter XX*

a. THE "LEADER'S" WORK

Let us for illustration take, $\frac{3}{5}$ of $\frac{10}{2}$ =.

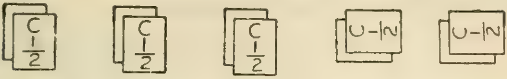
Under Chapter X the "leader's" construction and oral work would have been as follows:

("This tells me to take $\frac{10}{2}$, make them into five *equal* groups, show three, hide two, and then put the 'seen groups' together.")

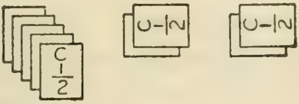
("I take $\frac{10}{2}$.")

(He takes a group of $\frac{10}{2}$.)

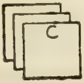
("I make them into five equal groups, show three, and hide two.")



("I now put the 'seen groups' together.")



Under Chapter XX this construction and the oral work would be supplemented by an exchange of $\frac{6}{2}$ for 3.



b. THE "WRITER'S" WORK

Under Chapter X the "writer's" complete written work would have been $\frac{3}{5}$ of $\frac{10}{2} = \frac{6}{2}$. Under Chapter XX this written statement would be extended as follows: $\frac{3}{5}$ of $\frac{10}{2} = \frac{6}{2} = 3$.

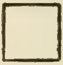
NOTE.—The aim of 3 is to show at this time, *before* the introduction of the *analysis* work of partition in *B*, what the construction and oral forms are.

B. The Analysis—Objective

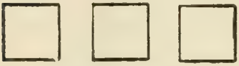
1. Its Basis

It is this "putting together" that the child must learn to analyze. In his work in applied problems he has doubtless been trained in diagraming analytical processes, the three classes of which are as follows:

a. Problems in the form of—

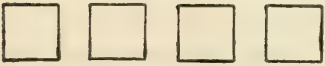


If one box holds 7 lbs.



3 boxes will hold 3×7 lbs. = 21 lbs.

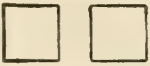
b. Problems in the form of—



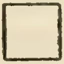
If 4 oranges cost 12¢.

1 will cost $\frac{1}{4}$ of 12¢ = 3¢.

c. Problems in the form of—



If 2 horses eat 12 qts. of grain



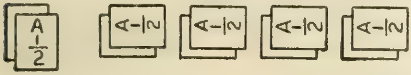
1 will eat $\frac{1}{2}$ of 12 qts. = 6 qts.

3 will eat 3 × 6 qts. = 18 qts.

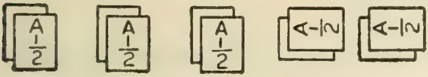
The child thus trained will easily recognize its application here and no special instruction will be needed. If he has not been so trained, time will have to be given to analysis development.

2. *The "Writer's" Diagram and Written Work*

The following is the "writer's" diagram and the form of his written work on the blackboard in the analysis of $\frac{3}{5}$ of $\frac{10}{2} =$, based on the work in applied problem analysis:



$\frac{1}{5}$ of $\frac{10}{2} = \frac{2}{2}$.



$\frac{3}{5}$ of $\frac{10}{2} = 3 \times$
$\frac{2}{2} = \frac{6}{2} = 3$.

NOTE.—The above, the diagram and the written work, is what the "*writer*" *always* puts on the blackboard in a class exercise in *analysis* when the class is working with *objects* or *diagrams*. If the class is *not* using objects or diagrams and the teacher does not call for a diagram, he puts on the *written* work *only*. No other directions for the "writer's" work in *analysis* exercises are needed.

If the "*leader*" is told to show the "*analysis* construction," his concrete work on his desk is in the form as shown in the diagram—the first construction showing one "seen group," the second showing the regular construction for the expression. The result of a partition solution should be called its quotient. Dividend is the proper term to apply to the quantity to be partitioned.

3. *The "Leader" in Analysis Work*

There are two forms in which the "leader" may show his objective analysis. These may be called the *full* form and the *short* form. Both forms should be taught, the short form for its own sake, the full form because it is best adapted for exercises in which the groups are mixed numbers.

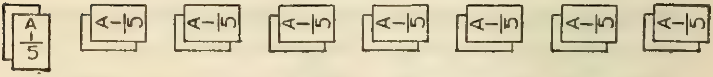
a. THE FULL FORM

The following shows the "leader's" construction and oral work:

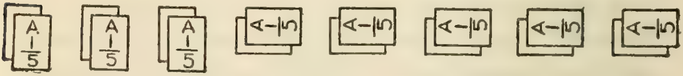
$\frac{3}{8}$ of $\frac{16}{5}$ =

("This tells me to take $\frac{16}{5}$, make them into eight *equal* groups, show three, hide five, and then put the 'seen groups' together. My dividend is $\frac{16}{5}$.")

("I made $\frac{1}{8}$ of $\frac{16}{5}$.")



("I now make $\frac{3}{8}$ of $\frac{16}{5}$.")

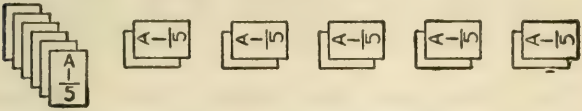


("One eighth of sixteen fifths is two fifths.")

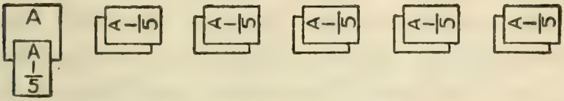
(As the leader says this he places his index finger on the "seen group," the $\frac{1}{8}$, in the first construction.)

("Three eighths of sixteen fifths are three times two fifths which are six fifths. My quotient is six fifths.")

(As the leader says this he puts the three "seen groups" together.)



("Six fifths is an improper fraction. So I exchange five fifths for one. I have one and one fifth.")

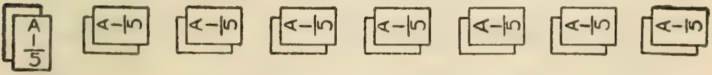


b. The Short Form

The following shows the "leader's" short form:

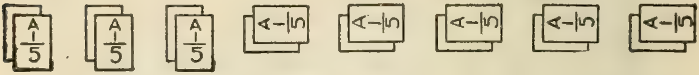
(It is unnecessary to repeat the "leader's" *oral* statements because the oral work for the two forms is one and the same—no difference whatever.)

When he says, "I make $\frac{1}{8}$ of sixteen fifths," he constructs—



as in the full form.

When he says, "I now make $\frac{3}{8}$ of sixteen fifths," he changes the first and second "*hidden*" groups to the "*seen*" form, making the construction—



He then puts the three "seen" groups together and changes to a mixed number in exactly the same way as in the full form.

The difference, therefore, between the full and the short forms is that in the former he actually constructs for each of the two steps in the analysis, while in the latter he constructs the first "full form" statement and then makes changes in this statement for his second analysis step.

4. Analysis Developed—Full Form

These should all be in "class exercise" form with a "leader" and a "writer."

a. INTRODUCTION

The instruction is given through an oral exercise.

"How many in $\frac{3}{4}$ of 8?"

("Six.")

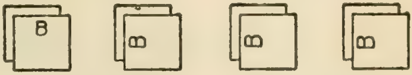
"Analyze it to show that it is six."

("¼ of eight is two. ¾ of eight is 3 times 2 which is 6.")

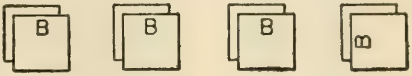
The teacher directs the "leader" (and those at their seats) to show both of the statements in the analysis with objects. She directs the "writer" to show the same with a diagram.

The "leader's" constructions are—

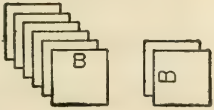
The ¼ of 8



The ¾ of 8

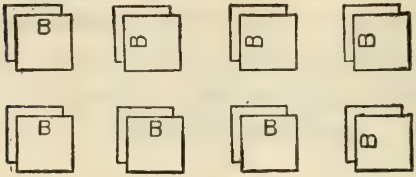


The ¾ of 8 =



The "writer's" diagram will show the first two statements only. The "writer" does not diagram the group which results from putting the "seen" groups together. The "writer" will, of course, write out the analysis as the leader gives it orally. Another exercise and then another are given in the same manner. (The teacher must refer to the constructions as "full form" constructions. The work should be called by teacher and pupil "analysis work." This will bring the terms "full form" and "analysis" into regular use.)

Each statement of the writer's analysis should be opposite the corresponding part of the diagram as follows:



$\frac{1}{4}$ of 8 = 2.

$\frac{3}{4}$ of 8 = 3
×2 = 6.

b. EXERCISES FOR CLASS USE

$\frac{2}{3}$ of 6 =
$\frac{4}{5}$ of 15 =
$\frac{5}{9}$ of $\frac{18}{2}$ =
$\frac{5}{8}$ of $\frac{16}{7}$ =
$\frac{11}{8}$ of $\frac{16}{12}$ =
$\frac{2}{7}$ of 14 =
$\frac{2}{3}$ of $\frac{15}{3}$ =
$\frac{2}{7}$ of $\frac{21}{4}$ =
$\frac{7}{5}$ of 15 =
$\frac{5}{4}$ of $\frac{20}{9}$ =
$\frac{3}{5}$ of $\frac{10}{7}$ =
$\frac{3}{8}$ of 24 =
$\frac{5}{6}$ of 12 =
$\frac{4}{3}$ of 9 =
$\frac{10}{7}$ of $\frac{21}{3}$ =

5. *Analysis Developed—Short Form*

This is entirely "class exercise" work with "leader" and "writer."

a. INTRODUCTION

The teacher will remember that in analysis work the "writer's" diagram and written forms are the same for the "short form" as for the "full form." (See Note under *2* above.)

This introduction, therefore, will consist in teaching the pupils how to make the short form *constructions* for the oral work. For illustration, in $\frac{3}{7}$ of $\frac{14}{3} =$, the construction would be that of $\frac{1}{7}$ of $\frac{14}{3} =$ for the *first* statement of the analysis, and then for the construction for the *second* statement he would merely change the first and second "unseen" groups to "seen" groups. This introduction is simple and the work is quickly brought into use.

b. Exercises for Class Use

These are like those for the "full form":

$\frac{3}{7}$ of $\frac{14}{5} =$
$\frac{9}{8}$ of $\frac{16}{3} =$
$\frac{7}{5}$ of $\frac{15}{7} =$
$\frac{10}{9}$ of $\frac{5}{3} =$
$\frac{11}{5}$ of $\frac{5}{8} =$
$\frac{7}{12}$ of $12 =$
$\frac{9}{7}$ of $\frac{7}{8} =$
$\frac{5}{2}$ of $\frac{8}{3} =$
$\frac{4}{3}$ of $15 =$

C. Written Work without Objects or Diagrams

1. *Class Exercises*

In these exercises one pupil and then another is sent to the blackboard to give the oral work as if with objects and also the work of the "writer" without the diagram.

Of course, there is no full form nor short form in this, the written work in analysis being given always in full statements as in *B* 2 above.

2. *Seat Analysis Work*

These are exercises in which each pupil writes out the work on paper using the analysis statements.

Dictations for Seat Work:

$\frac{3}{2}$ of $\frac{6}{7} =$ $\quad$ $\frac{8}{5}$ of $\frac{5}{2} =$ $\quad$ $\frac{10}{2}$ of $12 =$

$\frac{5}{3}$ of $9 =$ $\quad$ $\frac{7}{4}$ of $\frac{32}{9} =$ $\quad$ $\frac{3}{12}$ of $\frac{24}{5} =$

$\frac{2}{7}$ of $\frac{35}{3} =$ $\quad$ $\frac{7}{8}$ of $\frac{32}{3} =$ $\quad$ $\frac{8}{9}$ of $45 =$

$\frac{10}{7}$ of $\frac{36}{7} =$ $\quad$ $\frac{13}{7}$ of $\frac{21}{4} =$ $\quad$ $\frac{9}{8}$ of $\frac{40}{5} =$

D. Mental Solutions

Dictations like those in *B*, *4 b* above are given for oral solutions in analysis form. These oral solutions are in the same form as the written.

SUMMARY.

1. Object work.
   - *a.* A brief review of constructions and oral work in even partition of integers and fractions without analysis in preparation for the development of the constructions and written work of partition analysis. (*A*.)
   - *b.* Reviews or instruction in the diagram forms and the accompanying written analysis forms of the three types of problems illustrated in partition analysis. (*B*, *1*.)
   - *c.* The development of objective (or diagram) partition, analysis, and the written analysis statements. (*B*, *2* to *5*.)

(i) Long form. (*B*, *3 a* and *4*.)
(ii) Short form. (*B*, *3 b* and *5*.)

2. Work without objects.
   Exercises with full written analysis. (*C*.)
3. Mental solutions—full oral analysis. (*D*.)

Time required for the work of this chapter, including reviews of Chapters II–XXI, five days. Total time for Chapters II–XXII, 110 days.

CHAPTER XXIII

UNEVEN PARTITION

Topics:—The Constructions and the Oral and Written Language (Analytical). All-Fraction and Part-Fraction Forms of Partition

A. Exercises having One Extra Unit

1. Constructions without Solutions

The first difficulty that the pupil meets in uneven partition is that of constructions. These exercises in concrete expression are given that he may have experiences in this work unhampered by considerations of analysis or forms of statements.

a. Naming the denominator of the fraction after the construction.

The teacher dictates on the blackboard an expression, as $\frac{3}{4}$ of 9, and asks the pupils to construct it. The child will distribute the 8 into 4 equal groups but *may* be at a loss to distribute the extra unit. If the work in Section 84 of Vol. 1 * has been kept occasionally reviewed since it was given, the children will not be at a loss to know what to do. The child that hesitates *should not be told.* He should be left to work out his problem alone. The only assistance at all allowable would be to ask,

---

* See also Chapter X, *A* Note.

"What does the expression tell you?"

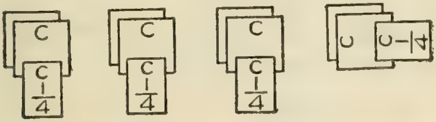
("It tells me to take 9, make them into 4 equal groups, etc.")

"What word do we emphasize in that?"

("Equal.")

"The groups must be *equal* and you must use all of your paper."

When the extra whole has finally been made into 4ths and properly distributed there are two questions * to be asked—



"Into what did you make your extra unit?"

("I made it into 4ths.")

"How many have you in each group?"

("I have $2\frac{1}{4}$.")

$\frac{2}{3}$ of 7.

The child will distribute the 6 and will doubtless make the extra whole into 3ds without hesitation.

---

* These questions must be asked after each construction in this step.

"Into what did you make the extra unit?"

("Into 3ds.")

"How many in each group?"

("$2\frac{1}{3}$.")

$\frac{5}{7}$ of 15.

The 14 are distributed and then the extra unit. The question as above must be asked and the pupil required to read one of his groups.

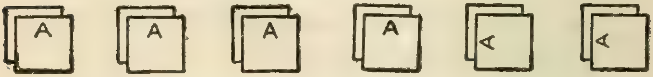
$\frac{4}{3}$ of 6.

(The pupil is assumed to know that $\frac{4}{3}$ of 6 means "all of the thirds of 6, and one of the thirds of another 6." (See Vol. I, Sections 111 and 113.) The teacher begins by asking,

"What does this mean?"

("This means all of the 3ds of 6 and one of the thirds of another 6.")

"You may construct."

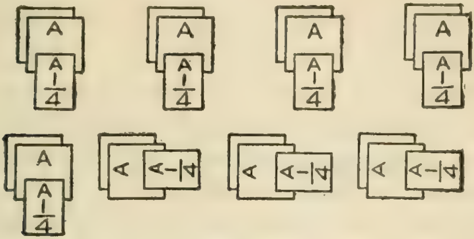


(If the pupils are not perfectly fresh in these even partition expressions, several exercises of this kind must be given before taking up such work in uneven partition. The question, "What does this mean?" must be asked with every dictation.)

$\frac{5}{4}$ of 9.

"What does this mean?"

("This means all of the 4ths of 9, etc.")
"You may construct."



(The constructions of this kind offer no greater difficulties than those in the exercises above.)

The questions, "Into what did you make your extra unit?" and "How many have you in each group?" must not be neglected.

Other similar exercises should follow:

$\frac{4}{5}$ of 11
$\frac{5}{6}$ of 13
$\frac{5}{4}$ of 13
$\frac{4}{3}$ of 7
$\frac{7}{9}$ of 19
$\frac{3}{4}$ of 1
$\frac{2}{3}$ of 4
$\frac{5}{4}$ of 9
$\frac{7}{8}$ of 9
$\frac{8}{6}$ of 1
$\frac{2}{3}$ of 13
$\frac{8}{7}$ of 15
$\frac{2}{3}$ of 1
$\frac{15}{9}$ of 10
$\frac{10}{7}$ of 8
$\frac{7}{3}$ of 10

The questions should follow each construction.

Following this, constructions should be given, in the same manner and with the same questions,* involving fractions whose numerators are 1.

---

*The word unit in the question, "Into what did you make your extra units," will, of course, refer to a *fractional* unit.

$\frac{3}{4}$ of $\frac{1}{2}$
$\frac{2}{3}$ of $\frac{1}{5}$
$\frac{2}{5}$ of $\frac{1}{3}$
$\frac{3}{2}$ of $\frac{1}{5}$
$\frac{5}{6}$ of $\frac{1}{4}$
$\frac{1}{4}$ of $\frac{1}{3}$
$\frac{7}{8}$ of $\frac{1}{2}$
$\frac{9}{7}$ of $\frac{1}{3}$
$\frac{1}{3}$ of $\frac{1}{6}$
$\frac{2}{5}$ of $\frac{1}{4}$
$\frac{4}{3}$ of $\frac{1}{2}$
$\frac{5}{6}$ of $\frac{1}{3}$
$\frac{7}{4}$ of $\frac{1}{8}$
$\frac{7}{3}$ of $\frac{1}{2}$
$\frac{6}{5}$ of $\frac{1}{4}$
$\frac{1}{9}$ of $\frac{1}{2}$

NOTE I.—The partition of fractions whose numerators are greater than unity ($\frac{3}{5}$ of $\frac{2}{7}$, $\frac{5}{7}$ of $\frac{4}{9}$, $\frac{3}{8}$ of $\frac{2}{9}$, etc.) must naturally follow the work with the partition of corresponding integral numbers ($\frac{3}{5}$ of 2, $\frac{5}{7}$ of 4, $\frac{3}{8}$ of 2, etc.). Partition work, however, like $\frac{2}{3}$ of $\frac{4}{2}$, $\frac{3}{5}$ of $\frac{6}{7}$, $\frac{5}{7}$ of $\frac{8}{9}$, etc., in which the numerator of the fraction to be divided is numerically greater than the denominator of the first term of the partition expression, offers, in construction work, difficulties so severe that the work must be made a later development step—a separate step.

b. Anticipating the denominator of the fraction.

In all work thus far, that under (*a*), the pupil has named the fraction which he made from the extra integral unit or from the single fractional unit *after* the construction. In this step, the teacher asks the *first* question, "Into what did you make the extra unit?" *before* the construction. The second question, "How many have you in each group?" is asked, as before, *after* the construction. However, the *first* question is necessarily modified slightly for this step. It is now, "Into what *must* you make your extra unit?" because of its being asked *in advance* of the construction. These exercises are like those in (*a*)—

$\frac{4}{5}$ of 11
$\frac{5}{6}$ of 19
$\frac{3}{7}$ of $\frac{1}{2}$
$\frac{9}{4}$ of $\frac{1}{3}$
$\frac{8}{7}$ of $\frac{1}{3}$
$\frac{9}{8}$ of 17
$\frac{11}{9}$ of $\frac{1}{2}$
$\frac{10}{3}$ of $\frac{1}{4}$
$\frac{5}{8}$ of 17
$\frac{7}{3}$ of 10
$\frac{5}{3}$ of $\frac{1}{6}$
$\frac{3}{7}$ of 15

2. *Constructions with Solutions*

a. Character of the Work

The exercises are like those in *1 a* above, excepting that in this step the solutions are an added feature. The dictations are with the sign =.

The work is objective, therefore the constructions must be in analysis form—some in "full form" the rest in the "short form."

The work must be in the form of *class exercises* with "leader" and "writer."

The "writer's" work must be in accordance with Chapter XXII. (*B*, 2, Note.)

The "leader" must state *in advance* of construction the denominator of the fraction to be made from the extra unit.

The "writer's" construction and oral work will, in general, follow the suggestions in Chapter XXII. (*B*, 2, Note.)

b. Illustrative of Construction and Oral Work

(1) The following is the "leader's" construction and oral work on $\frac{7}{5}$ of $\frac{1}{2}$ = (full form).

"This means that we are to make all the 5ths of $\frac{1}{2}$, two of the 5ths of another half, and then put them together."

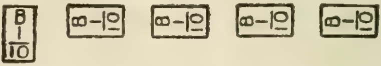
"We must first make $\frac{1}{5}$ of $\frac{1}{2}$ and then $\frac{7}{5}$ of $\frac{1}{2}$."

"I take $\frac{1}{2}$."

"We cannot make $\frac{1}{2}$ into five equal groups."

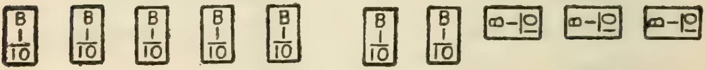
"I exchange $\frac{1}{2}$ for $\frac{5}{10}$."

"I make $\frac{1}{5}$ of $\frac{5}{10}$."



"I take another $\frac{1}{2}$ and exchange it for $\frac{5}{10}$, then another $\frac{1}{2}$ and exchange for $\frac{5}{10}$."

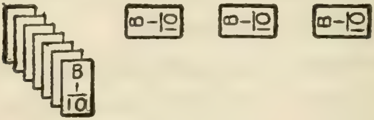
"I make $\frac{7}{5}$ of $\frac{5}{10}$."



"I put them together."

"One fifth of $\frac{5}{10} = \frac{1}{10}$."

"Seven fifths of $\frac{5}{10} = 7$ times $\frac{1}{10} = \frac{7}{10}$."



The "writer's" work as before stated is—

1. A diagram of $\frac{1}{5}$ of $\frac{1}{2}$ corresponding to the "leader's "construction above, and a diagram of $\frac{7}{5}$ of $\frac{1}{2}$ corresponding to the "leader's" construction.

2. The statement $\frac{1}{5}$ of $\frac{1}{2} = \frac{1}{5}$ of $\frac{5}{10} = \frac{1}{10}$ written opposite its diagram, and the statement $\frac{7}{5}$ of $\frac{1}{2} = 7 \times \frac{1}{10} = \frac{7}{10}$ * written opposite its diagram.

The "writer" makes no diagram of the groups "put together."

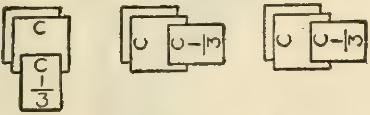
(1) The following illustrates the "leader's" work with $\frac{5}{3}$ of 7 = (full form).

"This means that we make all the 3ds of 7, two of the 3ds of another 7, and then put them together."

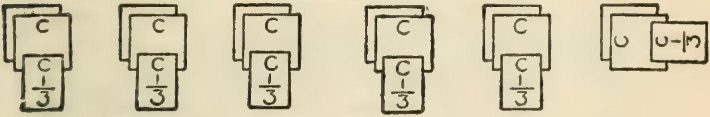
"We must first make $\frac{1}{3}$ of 7 and then $\frac{5}{3}$ of 7."

"Take 7."

"I must make $\frac{1}{3}$ of 6, then exchange 1 for $\frac{3}{3}$, and then complete the groups."



"I take two 7's and make $\frac{5}{3}$ of 7."



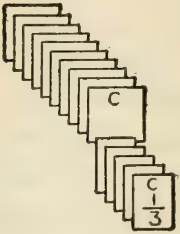
"I put them together."

"Five times $\frac{1}{3}$ are five thirds."

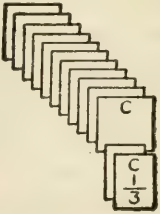
---

* This is the writer's form of statement in all exercises in which the "extra unit" is a fraction; as in $\frac{3}{7}$ of $\frac{1}{3}$ =, $\frac{4}{5} = \frac{1}{4}$, $\frac{5}{4}$ of $\frac{1}{6}$ =, etc.

"Five times 2 are ten."
"I have ten and five 3ds."



"This is an improper fraction."
"I exchange $\frac{3}{3}$ for 1."
"I have 11 and $\frac{2}{3}$."



The writer's work will show—

(i) A two statement diagram, and

(ii) A written statement opposite each part of the diagram.

$\frac{1}{3}$ of 7 $= 2\frac{1}{3}$
$\frac{5}{3}$ of 7 $= 5 \times 2\frac{1}{3} = 10\frac{5}{3} = 11\frac{2}{3}$.

(2) The "leader's" work in "short form" construction.

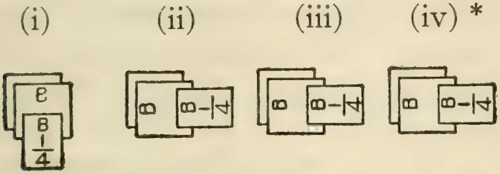
$\frac{3}{4}$ of 9 = (short form).

"This means that we must take 9, make them into 4 equal groups, show 3, and hide 1."

"Take 9."

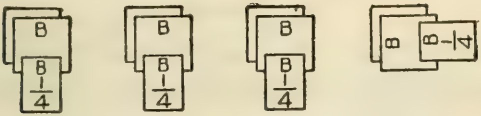
"I must make $\frac{3}{4}$ of 8, exchange the extra unit for $\frac{4}{4}$, and then complete the groups."

"I make first $\frac{1}{4}$ of 9."



"One fourth of 9 $=2\frac{1}{4}$."

"Three fourths of 9 $=3$ times $2\frac{1}{4}=6\frac{3}{4}$."



The pupil completes the oral work and the construction in the form as shown in *2 b 1* above.

The "writer's" work is the diagrams and statements as in *2 b 1* and *2*. The "writer's" work is the same for "short form" as for "long form."

---

* As the leader says, "I make first $\frac{1}{4}$ of 9," he makes this construction. Then with the first part "three fourths of 9" of his next statement he changes (ii) and (iii) of this construction to "seen" groups changing the construction.

(3) *Seat Work without Objects—Analysis Form*

This work is in two forms—

a. CLASS EXERCISES

The teacher sends one and then another pupil to the board to do the work of "leader" and "writer." This pupil gives the oral work the same as with the objects. His written work is the "writer's" two written statements as in *A* 2 *b* 2 above. The diagrams are not made, of course, because this is work *without objects.*

b. SEAT SOLUTIONS

These are exercises for practice in which the pupils write out the solutions on their papers in the "writer's" two-statements (analysis) form.

FORM OF THE EXERCISES

The dictations are like those suggested in *A* 1 *a* and *b* above excepting that the = is used with each – for solutions.

(4) *Mental Solutions—Analysis Form*

These are exercises like those in *A* 1 *a* and *b* (using =) for pupils to give the solutions in the two-statement (analysis) form—the same as that for written work. Mental work begins after the pupil's written work without objects has become somewhat rapid. The dictations should be mostly written.

The oral statement should be as brief as possible:—

$\frac{2}{3}$ of 7 =
$\frac{1}{3}$ of 7 is $2\frac{1}{3}$. $\frac{2}{3}$ of 7 = 2 times $2\frac{1}{3}$ or $4\frac{2}{3}$.
$\frac{2}{3}$ of $\frac{2}{5}$ =
$\frac{1}{3}$ of $\frac{2}{5}$ is $\frac{1}{3}$ of $\frac{6}{15}$ or $\frac{2}{15}$.
$\frac{2}{3}$ of $\frac{2}{5}$ is 2 times $\frac{2}{15}$ or $\frac{4}{15}$.
$\frac{7}{6}$ of $\frac{1}{2}$ =
$\frac{1}{6}$ of $\frac{1}{2}$ is $\frac{1}{6}$ of $\frac{6}{12}$ or $\frac{1}{12}$.
$\frac{7}{6}$ is 7 times $\frac{1}{12}$ or $\frac{7}{12}$.
$\frac{8}{2}$ of $\frac{1}{3}$ =
$\frac{1}{2}$ of $\frac{1}{3}$ is $\frac{1}{2}$ of $\frac{2}{6}$ or $\frac{1}{6}$.
$\frac{8}{2}$ is 8 times $\frac{1}{6}$ or $\frac{8}{6}$ or $1\frac{1}{3}$.

Very many mental exercises should be given.

B. Exercises having Two or More Extra Units *

1. *Constructions without Solutions*

a. AIM.—The work of this step is like that in (*A*) excepting that there the unit that for the grouping had to be made into a fraction (or into a higher fraction as in the case of the partition of a fraction) was naturally exchanged for fractional units having the same denominator as the partitioning fraction, while here that denomination would often be too high—the rule of construction being that the exchange must be for the largest (lowest) fractional units that will meet the requirements of the partition.

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* These exercises include the partition of *integral* numbers of all kinds whether with 2, 3, 4, 5, or any other number of extra units (units that can be partitioned only by changing them to fractions); and *fractional* numbers having numerators of 2, 3, 4, 5, or any other number of units, *provided* that this number is *smaller* than the *denominator* of the partitioning fraction. Take, for illustration, the expression $\frac{3}{5}$ of $\frac{2}{7}$ =.

If we give the child a partition exercise like $\frac{3}{4}$ of 6, he will proceed at once, after his experiences in (*A*), to construct $\frac{3}{4}$ of 4, then to make the 2 into 4ths, and then to complete the

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The numerator (2) of the number to be partitioned is numerically smaller than the denominator (5) of the partitioning fraction. This expression, therefore, is one that may be used in this step. On the other hand, in the expression $\frac{2}{3}$ of $\frac{4}{5} =$, the numerator (4) of the number is numerically larger than the denominator (3) of the partitioning fraction. The expression, therefore, is one that may *not* be used in this step.

The difficulties in this step in partition are wholly in the constructions. Integral numbers of two or more units, and fractional numbers having numerators that are less than the denominator of the partitioning fraction, offer one and the same kind of construction difficulty; while in fractions having numerators larger than the denominator of the partitioning fraction the child is confronted with a new problem in construction. Hence two development steps.

groups with the eight 4ths. Now, while this is a true partition, it is very loose construction and, as number development work, must not be tolerated. The extra units could have been made into larger units (halves) for the completion of the group. Halves being larger units this construction would be correct, the former faulty.

The aim of this section is to teach the child that in a partition involving fractions the construction is faulty if the fraction units put into the groups are not in the lowest terms consistent with the partition requirements.

b. Method of Presentation

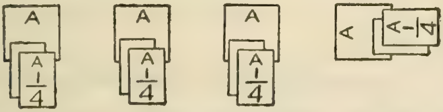
(i) The teacher writes on the blackboard $\frac{3}{4}$ of 6.

"How many extra units in this expression?"

("Two.")

"You may construct it."

The construction will be—



"You have used higher fractions than necessary."

"What is the series of fractions from 2?"

("It is $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, $\frac{10}{5}$, etc.")

"You used $\frac{8}{4}$. Could you have made a lower fraction than $\frac{8}{4}$ into 4 equal groups?"

("I could have used $\frac{4}{2}$.")

"You used the wrong fraction, then. We must always use the lowest one possible."

"You may put your objects back and construct again."

(ii) $\frac{5}{6}$ of 8.

"How many units to be made into fractions?"

("Two.")

"You must tell me in advance what you will make the 2 into."

"Give me the fraction series from 2."

("The series is $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, $\frac{10}{5}$, $\frac{12}{6}$, etc.")

"What is the lowest term that you can use?"

("I must use $\frac{6}{3}$.")

"You may construct."

The pupils distribute the 6, then exchange the 2 for $\frac{6}{3}$, then complete the groups.

"How many in your first group?"

("$1\frac{1}{3}$.")

(iii) $\frac{5}{8}$ of 4.

"Can you make 4 into 8 equal groups?"

"Into what must you make the 4?"

("Into halves.")

"Which term of the fraction series in this?"

("The first.")

"You may construct."

(The constructions are made.)

"How many in the first group?"

("*One* half.")

(iv) $\frac{3}{4}$ of $\frac{2}{3}$.

"What is the series of equivalents of $\frac{2}{3}$?"

("$\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$, etc.")

"Which must we use for our partition?"

("The first, $\frac{4}{6}$, because it is the lowest.")

"You may construct."

(The pupils exchange the $\frac{2}{3}$ for $\frac{4}{6}$ and construct.)

"How many in the first group?"

("*One* sixth.")

(v) $\frac{2}{3}$ of $\frac{2}{3}$.

"Can you make $\frac{2}{3}$ into 3 equal groups?"

"Into what must you change it?"

("Into $\frac{6}{9}$.")

"Which term of the series of higher equivalents is $\frac{6}{9}$?"

("The third.")

"You may construct."

"How many in the first group?"

("$\frac{2}{9}$.")

c. THE ORAL WORK

The teacher develops the oral work by choosing one, then another, and then another, to stand at his desk and "talk" as he constructs—the other pupils constructing as his oral work proceeds.

(i) The teacher dictates on the blackboard an exercise as $\frac{2}{3}$ of 11. The "leader" is chosen.

("This means that we make 11 into 3 groups and show 2. I take 11. I must make $\frac{2}{3}$ of 9 and then $\frac{2}{3}$ of 2. I make $\frac{2}{3}$ of 9.")

He now makes the $\frac{2}{3}$ of 9, all the other pupils doing the same.

("I cannot make 2 into 3 equal groups so I *think* * the 2 into $\frac{4}{2}$. I cannot make $\frac{4}{2}$ into 3 equal groups so I think the 2 into $\frac{6}{3}$. I can use the $\frac{6}{3}$ and complete the groups.")

"How many in your first group?"

("$3\frac{2}{3}$.")

(ii) $\frac{2}{3}$ of $\frac{2}{5}$.

The "leader" is selected.

("This means that we make $\frac{2}{5}$ into 3 equal groups and show 2. I cannot make $\frac{2}{5}$ into 3 equal groups, so I *think* it into $\frac{4}{10}$. I cannot make $\frac{4}{10}$ into 3 equal groups, so I think the

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* The teacher may prefer, for a few exercises, to have the pupil actually exchange 2 for $\frac{4}{2}$ and decide from the concrete expression that the partition cannot be made, then to put back the $\frac{4}{2}$ and exchange the 2 for $\frac{6}{3}$. It seems hardly necessary, however, after the child has had so much fraction experience, to have him make such actual exchange. To *think* the exchange is simply the series work of Chapter XVII—work that was carefully developed objectively and with which he should have been kept familiar by occasional exercises since.

$\frac{2}{5}$ into $\frac{6}{15}$. I can use $\frac{6}{15}$. I exchange my $\frac{2}{5}$ for $\frac{6}{15}$ and make the groups. I show 2 and hide 1.")

This illustrates the form of the oral work. Very many exercises of this kind should be given in the partition of both integral and fractional numbers.

d. EXERCISES FOR ORAL WORK

$\frac{2}{3}$ of 2
$\frac{3}{4}$ of $\frac{3}{2}$
$\frac{5}{6}$ of 14
$\frac{7}{5}$ of 12
$\frac{8}{7}$ of 17
$\frac{4}{7}$ of $\frac{6}{10}$
$\frac{9}{10}$ of $\frac{5}{7}$
$\frac{11}{12}$ of $\frac{8}{9}$
$\frac{7}{3}$ of $\frac{2}{5}$
$\frac{4}{5}$ of 13
$\frac{2}{3}$ of 5
$\frac{5}{7}$ of $\frac{2}{3}$
$\frac{3}{5}$ of $\frac{4}{7}$
$\frac{5}{8}$ of $\frac{5}{6}$
$\frac{1}{4}$ of $\frac{2}{9}$
$\frac{7}{8}$ of 13
$\frac{6}{12}$ of $\frac{2}{5}$
$\frac{7}{10}$ of $\frac{4}{9}$
$\frac{11}{12}$ of 16
$\frac{2}{9}$ of $\frac{3}{7}$

2. *Constructions with Solutions—Analysis Form*

These exercises are like those in *1 c* and *d* above excepting that the sign is added to indicate a solution—the putting together of the "seen" groups. The work being objective must be in *analysis form.*

a. CLASS EXERCISES WITH "LEADER" AND "WRITER"

The illustrations of the leader's "full form" and "short form" work and the writer's diagram and written work in *A*, 2 *b* above suffice for the corresponding work here. This step, apart from the series work to determine the lowest denominator, is in all respects very much like that. The work here should follow that outline.

b. Forms of Written Work

The following solutions show the writer's work and the forms of the seat work without objects in *3* below—

$\frac{5}{8}$ of $\frac{5}{6}$ =
$\frac{1}{8}$ of $\frac{5}{6} = \frac{1}{8}$ of $\frac{40}{48} = \frac{5}{48}$
$\frac{5}{8}$ of $\frac{5}{6} = 5 \times \frac{5}{48} = \frac{25}{48}$

$\frac{4}{5}$ of 2 =
$\frac{1}{5}$ of $2 = \frac{1}{5}$ of $\frac{10}{5} = \frac{2}{5}$
$\frac{4}{5}$ of $2 = 4 \times \frac{2}{5} = \frac{8}{5} = 1\frac{3}{5}$

$\frac{3}{7}$ of 10 =
$\frac{1}{7}$ of $10 = \frac{1}{7}$ of $7\frac{21}{7} = 1\frac{3}{7}$
$\frac{3}{7}$ of $10 = 3 \times 1\frac{3}{7} = 3\frac{9}{7} = 4\frac{2}{7}$

3. Seat Work without Objects—Analysis Form

The suggestions for this work are the same as for the corresponding work in *A*, *3 a* and *b* above and need not be repeated here. For forms of written work see *2*, *b* above.

4. Mental Solutions—Analysis Form

The teacher is referred to *A*, *4* above for full suggestions for this work.

5. Exercises for Class Use

The following are exercises for use under *2*, *3* and *4* above:—

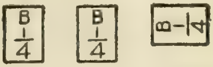
$\frac{3}{8}$ of 12 =
$\frac{5}{7}$ of $\frac{2}{3}$ =
$\frac{5}{8}$ of $\frac{5}{6}$ =
$\frac{5}{3}$ of $\frac{2}{9}$ =
$\frac{5}{4}$ of 15 =
$\frac{7}{9}$ of $\frac{6}{7}$ =
$\frac{5}{12}$ of $\frac{8}{10}$ =
$\frac{11}{12}$ of $\frac{2}{3}$ =
$\frac{7}{8}$ of $\frac{4}{5}$ =
$\frac{7}{5}$ of 9 =
$\frac{3}{4}$ of $\frac{3}{8}$ =
$\frac{2}{9}$ of $\frac{3}{8}$ =
$\frac{2}{15}$ of $\frac{3}{6}$ =
$\frac{8}{9}$ of 12 =
$\frac{7}{8}$ of 14 =
$\frac{8}{7}$ of 10 =
$\frac{5}{3}$ of 8 =
$\frac{7}{4}$ of $\frac{3}{11}$ =
$\frac{7}{6}$ of 15 =
$\frac{6}{5}$ of $\frac{1}{2}$ =

C. Exercises with Miscellaneous Fractions

1. *Constructions without Solutions*

a. AIM.—This step has to do with the partition of fractions whose numerators are numerically larger than the denominator of the partitioning fraction * like $\frac{2}{3}$ of $\frac{4}{2}$. It offers the child his first real construction problem. In constructing $\frac{2}{3}$ of 5 he constructed $\frac{2}{3}$ of 3 and then by exchanging the 2 for $\frac{6}{3}$ completed the groups. In the same manner in $\frac{5}{8}$ of 10 he constructed $\frac{5}{8}$ of 8, exchanged the 2 for $\frac{16}{8}$, and completed the partition.

When in this step he is given a dictation like $\frac{2}{3}$ of $\frac{5}{4}$, he will almost certainly construct the $\frac{2}{3}$ of $\frac{3}{4}$ and then, after exchanging the $\frac{2}{4}$ for $\frac{6}{12}$,



attempt to complete the groups. The moment he attempts to put the $\frac{6}{12}$ into the groups with the 4ths he will discover that he is constructing an "impossible" number. He will begin to realize at once that $\frac{2}{3}$ of 5 and $\frac{2}{3}$ of $\frac{5}{4}$ are two very different constructions and that he has a *problem* before him—a problem of a construction from $\frac{2}{3}$ of $\frac{5}{4}$ that will not be "impossible."

The teacher should leave the pupils to master the difficulty alone, each for himself.

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* Compare foot-note, page 361,

When a pupil finds that his exchange and construction are wrong, he should be told to exchange back for his original number (concrete) and try again.

He will solve the problem if left to his own resources. The second attempt will be successful in most cases.

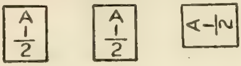
b. Method of Presentation

This is in the form of a "class exercise" in which each child works at constructions independently. The teacher writes a dictation on the blackboard—

(i) $\frac{2}{3}$ of $\frac{4}{2}$.

"You may construct it."

As stated in *a*, the pupils will construct

$\frac{2}{3}$ of $\frac{3}{2}$  and then, having

exchanged $\frac{1}{2}$ for $\frac{3}{6}$, will attempt to complete the groups with 6ths.

As one and then another pupil indicates by look and action that his construction will not "go together," it would be well for the teacher to suggest to the individual that he would better exchange back and try again. In a few minutes all will have discovered that this is their old problem of finding the proper term of the series of equivalents of $\frac{4}{2}$ ($\frac{12}{6}$) and construct with all one kind of units.

(ii) $\frac{5}{6}$ of $\frac{8}{9}$.

The same plan is to be followed with this. Very few will repeat the error of the work on (*1*).

(iii) $\frac{7}{8}$ of $\frac{10}{12}$.

The same plan again. All will construct this at once.

(iv) $\frac{4}{5}$ of 7.

This is to bring the pupil back to the work with integers. He must find $\frac{4}{5}$ of 5 and complete the groups with $\frac{10}{5}$.

(v) $\frac{3}{4}$ of $\frac{2}{3}$.

This brings him back to the fraction partition in (*B*).

(vi) $\frac{3}{5}$ of $\frac{7}{8}$.

This is (*1*) repeated. It brings the child back to his new problem.

c. MIXED EXERCISES FOR CONSTRUCTION

Addition exercises should be given before the oral work (*d*) is attempted. These must be mixed *integral* and *fractional* exercises:—

$\frac{3}{4}$ of $\frac{7}{12}$
$\frac{2}{3}$ of 2
$\frac{5}{6}$ of $\frac{8}{9}$
$\frac{5}{6}$ of 8

$\frac{4}{5}$ of 6
$\frac{4}{5}$ of $\frac{6}{3}$
$\frac{5}{8}$ of $\frac{10}{3}$
$\frac{4}{9}$ of 6

$\frac{2}{5}$ of $\frac{4}{10}$
$\frac{7}{8}$ of 10
$\frac{7}{8}$ of $\frac{10}{4}$
$\frac{4}{3}$ of 5

$\frac{9}{10}$ of $\frac{11}{2}$
$\frac{9}{10}$ of 12
$\frac{5}{12}$ of 18
$\frac{5}{12}$ of $\frac{18}{7}$

d. THE ORAL WORK

This is exactly like that in *B*, *1 c* (*i*) and (*ii*) above and need not be repeated here.

e. Exercises for Oral Work

$\frac{3}{4}$ of $\frac{7}{8}$
$\frac{2}{3}$ of 5
$\frac{3}{8}$ of $\frac{10}{12}$
$\frac{2}{5}$ of $\frac{8}{2}$
$\frac{6}{7}$ of $\frac{3}{2}$
$\frac{5}{7}$ of 8
$\frac{6}{9}$ of 12
$\frac{3}{8}$ of $\frac{12}{7}$
$\frac{2}{3}$ of $\frac{1}{2}$
$\frac{5}{8}$ of 3
$\frac{5}{7}$ of 9
$\frac{7}{12}$ of $\frac{3}{5}$
$\frac{11}{12}$ of 15
$\frac{11}{12}$ of $\frac{15}{4}$
$\frac{7}{8}$ of $\frac{10}{3}$
$\frac{7}{9}$ of 10

2. *Constructions with Solutions—Analysis Form*

This work is in all respects like that in *B 2* above and the suggestions as to the character and form of the work and as to "Exercises with Leader and Writer" apply here. It is not necessary to repeat them. The exercises in *1 c* and *e* above may be used here by adding the solution sign (=) to each.

3. *Seat Work without Objects—Analysis Form*

For suggestions on this work see *A*, *3*, *a* and *b;* and *B*, *2* above. For exercises see (5) below.

4. *Mental Solutions*

For full suggestions applicable to this, see *A*, *4* above. For exercises, see (5) below. The dictations, at least most of them, should be written. By changing them to solution form (adding =), the exercises in *1 c* and *e* above may be used.

5. *Exercises for Class Use**

$\frac{2}{5}$ of $\frac{8}{9}$ =
$\frac{3}{4}$ of 7 =
$\frac{5}{8}$ of $\frac{11}{12}$ =
$\frac{5}{12}$ of $\frac{8}{5}$ =
$\frac{4}{5}$ of 2 =

$\frac{7}{9}$ of $\frac{5}{2}$ =
$\frac{5}{3}$ of $\frac{2}{5}$ =
$\frac{9}{10}$ of $\frac{5}{7}$ =
$\frac{3}{8}$ of 12 =
$\frac{5}{9}$ of 11 =

$\frac{3}{4}$ of $\frac{5}{6}$ =
$\frac{4}{5}$ of $\frac{6}{7}$ =
$\frac{5}{6}$ of $\frac{7}{8}$ =
$\frac{6}{7}$ of $\frac{8}{3}$ =
$\frac{7}{8}$ of $\frac{10}{7}$ =

$\frac{5}{8}$ of $\frac{8}{3}$ =
$\frac{7}{9}$ of $\frac{3}{4}$ =
$\frac{5}{6}$ of $\frac{9}{10}$ =
$\frac{2}{3}$ of $\frac{5}{8}$ =
$\frac{3}{2}$ of $\frac{7}{8}$ =

D. Exercises in Part-Fraction and All-Fraction Partition

1. *Constructions with Solutions—Analysis Form*

a. AIM.—This step is to introduce these terms and to give the pupil experience in the partition in fraction form of integral numbers whose numerical value is greater than that of the denominator of the partitioning fraction.

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*Most of the pupils will doubtless have discovered by this time that the result of a solution may be anticipated by taking the products of the numerators and the denominators. To test the pupils in this matter the teacher should ask: Can you tell me now in advance what the result of this solution will be? If the answers indicate that they have made the discovery, she should hereafter go through the lists of the exercises for each day's work in advance of the regular solutions and ask this question upon each of the problems. The pupils after this exercise will proceed to the work of the solutions, written or mental, in the regular way. This is a preparation for the short processes later—Chapter XXVII.

It is simple, practically involves no new form of construction, and is easily learned. In $\frac{3}{4}$ of 5, the pupil has been constructing $\frac{3}{4}$ of 4, then exchanging 1 for 4ths, and then completing the groups. That form of construction we shall call part-fraction work. The aim in this section is to teach him another way of partitioning such an expression—by changing all to fractional form, exchanging, in this case, 5 for 4ths, and constructing the groups wholly with the fractions. In fact, it gives the same form to the construction of $\frac{4}{5}$ of 7 as to the construction of $\frac{4}{5}$ of $\frac{7}{2}$.

The pupil will be taught to speak of these two forms of the partition of integral numbers as *part-fraction* form and *all-fraction* form.

b. CLASS EXERCISES WITH "LEADER" AND "WRITER"

(i) $\frac{2}{3}$ of $\frac{5}{2}$ =

The leader and writer go through the construction and oral work as usual (*A*, *2 b 1*) and reduce the result to lowest terms.

"What did you exchange?"

(This question is to apply only to the exchange for the *partition* and not to that which reduces to lowest terms.)

("I exchanged all for fractions.")

(The writer's written work is not to be erased. It is left on the blackboard.

(ii) $\frac{3}{4}$ of 2 =

The leader and writer go through the construction and oral work as before.

"What did you exchange?"

("I exchanged 2 for $\frac{8}{4}$.")

"Did you exchange all or part of the number for fractions?"

("I exchanged all for fractions.")

(The writer's work is left on the blackboard.)

(iii) $\frac{5}{6}$ of 8 =

The leader's and the writer's work are followed, as before, by the questions—

"What did you exchange?"

("I exchanged 2 for 3ds.")

"Did you exchange all or part of the number for fractions?"

("I exchanged only a part for fractions.")

"In $\frac{2}{3}$ of $\frac{5}{2}$ =and in $\frac{3}{4}$ of 2 = you exchanged all for fractions. In $\frac{5}{6}$ of 8 = you exchanged only a part. This was proper, but I want you to take $\frac{5}{6}$ of 8 again and see if you can partition by exchanging *all* for fractions."

(iv) $\frac{3}{4}$ of 6 =

The leader and the writer do their work as before.

"What did you exchange?"

("2 for $\frac{4}{2}$.")

"Did you exchange all or only a part of the number for fractions?"

("I exchanged only a part for fractions.")

"You may construct $\frac{3}{4}$ of 6 = "*all-fraction.*"

(v) $\frac{7}{8}$ of 2 =

The work of the leader and writer as before. The questions are asked, the pupil replying that he used the *all-fraction* form.

"You may construct it by exchanging only a *part* of it for a fraction."

This the pupils will decide to be impossible.

(vi) $\frac{5}{8}$ of $\frac{10}{3}$ =

Following the plan as in (5), asking the pupils after their construction to try the other form, they will decide that this construction is possible only by an *all-fraction* partition. The pupils must be required to use the terms all-fraction and part-fraction wherever they may be applied.

(vii) $\frac{7}{8}$ of 4 =

Following the same plan, asking the pupils after their construction to try the other form, they will decide that this construction can be made only by *all-fraction* partition. The pupils use the terms.

(viii) $\frac{4}{5}$ of 7 =

Following the same plan, the pupils will find it possible to construct by *part-fraction* and also by *all-fraction* partition. The pupils should so state—using the terms.

This work is to be continued until the pupils are able to state *in advance* which form of partition should be used and whether the other form is or is not possible.

c. Exercises for Class Use

These are to be used with leader and writer for work as in (*b*).

$\frac{4}{5}$ of 11 =
$\frac{2}{3}$ of $\frac{2}{5}$ =
$\frac{3}{7}$ of $\frac{5}{8}$ =
$\frac{7}{8}$ of $\frac{10}{12}$ =
$\frac{5}{9}$ of $\frac{12}{7}$ =
$\frac{8}{9}$ of 2 =
$\frac{7}{12}$ of 15 =
$\frac{5}{11}$ of $\frac{5}{3}$ =
$\frac{5}{12}$ of $\frac{16}{9}$ =
$\frac{5}{12}$ of 16 =
$\frac{4}{3}$ of 2 =
$\frac{4}{3}$ of $\frac{2}{5}$ =
$\frac{6}{7}$ of 8 =
$\frac{6}{7}$ of $\frac{8}{9}$ =
$\frac{5}{12}$ of 3 =
$\frac{5}{12}$ of 18 =
$\frac{5}{12}$ of $\frac{18}{3}$ =
$\frac{5}{6}$ of 8 =
$\frac{5}{6}$ of $\frac{8}{5}$ =
$\frac{5}{6}$ of 3 =

2. *Seat Work without Objects—Analysis Form*

This is work that in (*1*) above which the pupils do on paper at their seats—practice work in all-fraction and part-fraction partition.

In these exercises the pupil writes out the solutions and writes opposite each whether the partition was in all-fraction or part-fraction form.

Each exercise that can be solved in either of the two forms is given *two* solutions, one in each form. This is a test of the pupil's grasp of the subject.

Very many exercises of this kind should be given. Pupils should learn to know instantly whether a problem admits of only one or of both forms of partition.

3. Mental Solutions

After the pupils have mastered the step in written solutions, the teacher should begin to give written dictations for mental solutions, the pupil stating the form of his partition in each case and giving *two* solutions, as in (2), of those exercises which admit of either form. The teacher must seek for brevity in this mental work.

4. Exercises for Class Use

$\frac{1}{5}$ of 4 =
$\frac{4}{6}$ of 8 =
$\frac{5}{9}$ of $\frac{12}{2}$ =
$\frac{7}{9}$ of $\frac{15}{8}$ =
$\frac{4}{8}$ of $\frac{5}{6}$ =

$\frac{7}{9}$ of $\frac{2}{3}$ =
$\frac{6}{5}$ of $\frac{12}{10}$ =
$\frac{12}{8}$ of $\frac{16}{14}$ =
$\frac{6}{5}$ of $\frac{7}{2}$ =
$\frac{2}{3}$ of 8 =

$\frac{7}{12}$ of 15 =
$\frac{11}{12}$ of $\frac{15}{4}$ =
$\frac{3}{12}$ of $\frac{6}{4}$ =
$\frac{5}{11}$ of $\frac{2}{7}$ =
$\frac{8}{2}$ of $\frac{7}{6}$ =

$\frac{3}{3}$ of $\frac{5}{2}$ =
$\frac{5}{3}$ of 5 =
$\frac{4}{2}$ of 6 =
$\frac{4}{2}$ of $\frac{6}{7}$ =
$\frac{8}{9}$ of $\frac{12}{13}$ =

E. Exercises with Mixed Numbers

1. Oral Work—without Objects

a. AIM.—This step is for experience in the partition of mixed numbers, the pupils using the *all-fraction* form with *every* exercise and giving a *second* solution in *part-fraction* form in each exercise that admits of part-fraction partition. Emphasis is thus laid upon the two forms of construction in order that as a result of the experiences in this step the pupil will become so familiar with both of them that in later work he will, in problems which admit of partition in either form, choose now one now the other according as the problem lends itself better to the one or to the other.

b. Objective Work Not Necessary

The child has now so much experience in partition and other constructions that we may safely rely upon his imagery being so well developed that he will be able to "*think*" the objective work with perfect clearness. Moreover, this plan is almost necessary because the *all-fraction* partition construction of mixed numbers easily involves fractions with large denominators. Objective work with large numbers should not be given unless it is positively necessary. Imagery may be developed with small numbers and it is *imagery* not number *facts* that fraction work with objects aims for.

The oral work of this step will be exactly the same as if objects were in use. The writer's work will, of course, be *without* the diagram.

c. Blackboard Oral and Written Work

This work is conducted by one pupil and then another being sent to the blackboard with an exercise to "talk" and write—doing the work of leader and writer.

(i) $\frac{2}{3}$ of $1\frac{1}{2} =$

A pupil is selected to go to the blackboard to talk and write.

"This tells me to partition $1\frac{1}{2}$ into 3 groups and show 2." I will use the all-fraction form. The part-fraction form can not be used.

"I *think* the $1\frac{1}{2}$ unto its higher series—$\frac{3}{2}$, $\frac{6}{4}$, $\frac{9}{6}$, etc.

"I can use $\frac{3}{2}$.

$\frac{1}{3}$ of $1\frac{1}{2}$ is $\frac{1}{3}$ of $\frac{3}{2}$ or $\frac{1}{2}$ (oral).

"$\frac{2}{3}$ of $1\frac{1}{2}$ is 2 times $\frac{1}{2}$ of $\frac{2}{2}$ (oral).

"$\frac{2}{3}$ of $1\frac{1}{2} = 2 \times \frac{1}{2} = \frac{2}{2} = 1$ (written under the dictation).

"I exchange $\frac{2}{2}$ for 1" (oral).

NOTE.—After stating and writing the *second analysis* statement the pupil writes the partition result (quotient) after the = in the original problem.

(ii) $\frac{3}{4}$ of $5\frac{2}{3} =$

A pupil is selected for the work at the blackboard.

"This tells me to partition $5\frac{2}{3}$ into 4 groups and show 3.

"I will use the part-fraction partition.

"$\frac{1}{4}$ of 4 is 1.

"The higher series from $1\frac{2}{3}$ is $\frac{5}{3}$, $\frac{10}{6}$, $\frac{15}{9}$, $\frac{20}{12}$, etc.

"I can use the 4th term—$\frac{20}{12}$.

"$\frac{1}{4}$ of 4 is 1.

"$\frac{1}{4}$ of $\frac{20}{12}$ is $\frac{5}{12}$.

"Therefore, $\frac{1}{4}$ of $5\frac{2}{3} = 1\frac{5}{12}$.

"$\frac{3}{4}$ of $5\frac{2}{3}$ is 3 times $1\frac{5}{12}$ or $3\frac{15}{12}$.

"I exchange $\frac{12}{12}$ for 1, and $\frac{3}{12}$ for $\frac{1}{4}$. I have $4\frac{1}{4}$."

The pupil's written work on the blackboard as he completes the oral work is as follows—

$\frac{3}{4}$ of $5\frac{2}{3} = \ldots\ldots\ldots$part-fraction.
$\frac{1}{4}$ of $4 \quad = \ldots\ldots\ldots 1$
$\frac{1}{4}$ of $1\frac{2}{3} = \frac{1}{4}$ of $\frac{20}{12} = \quad \frac{5}{12}$
$\therefore \frac{1}{4}$ of $5\frac{2}{3} = \ldots\ldots \quad 1\frac{5}{12}$*
$\frac{3}{4}$ of $5\frac{2}{3} = 3 \times 1\frac{5}{12} = 3\frac{15}{12} = 4\frac{1}{4}$.

The problem above should now have this written after it to show its answer—$\frac{3}{4}$ of $5\frac{2}{3} = 4\frac{1}{4}$.

When in his oral work he says, "$\frac{1}{4}$ of 4 is 1," his written quotient begins to take form $\frac{1}{4}$ of $5\frac{2}{3} = 1$.

When he says, "$\frac{1}{4}$ of $\frac{20}{12}$ is $\frac{5}{12}$," the statement becomes $\frac{1}{4}$ of $5\frac{2}{3} = 1\frac{5}{12}$.

Inasmuch as this problem admits of all-fraction solution the pupil at the board should at once proceed to that form of solution.

("This tells me, etc.")

"I will use the all-fraction partition.

"I will think the series of higher equivalents of $5\frac{2}{3}$" (or he may say, "I will write them on the board.")

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* The sign $\therefore$ means therefore.

"They are $\frac{17}{3}$, $\frac{34}{6}$, $\frac{51}{9}$, $\frac{68}{12}$, etc.

"I can use $\frac{68}{3}$.

"$\frac{1}{4}$ of $15\frac{2}{3}$ is $\frac{1}{4}$ of $\frac{68}{12}$ or $\frac{17}{12}$ or $1\frac{5}{12}$.

"$\frac{3}{4}$ of $15\frac{2}{3}=3$ times $1\frac{5}{12}$ or $3\frac{15}{12}$ or $3\frac{1}{4}$."

His written work on the board would be—

$\frac{3}{4}$ of $5\frac{2}{3}=$all-fraction.

$\frac{1}{4}$ of $5\frac{2}{3}=\frac{1}{4}$ of $\frac{68}{12}=\frac{17}{12}=1\frac{5}{12}$.

$\frac{3}{4}$ of $5\frac{2}{3}=3\times\frac{15}{12}=3\frac{15}{12}=1\frac{1}{4}$.

These illustrations suffice for the written and oral work in mixed number partition.

2. *Seat Work without Objects*

No special suggestions are needed for this work. The teacher writes the dictations on the blackboard for the pupils to solve on their papers.

It is advisable to require at first that the pupil make *two* solutions of all dictations that admit of part-fraction and all-fraction partition. Afterwards the child should choose the form of his partition unless the dictation specifies one or the other form. If the teacher finds that pupils are neglecting the part-fraction form she should specify this form very frequently. *It is the practical form.*

The form of the pupil's work in these written solutions should be that given in *1 d* above.

EXERCISES FOR CLASS USE

$\frac{3}{8}$ of $3\frac{1}{2} =$	$\frac{1}{6}$ of $19\frac{1}{2} =$	$\frac{5}{3}$ of $2\frac{1}{8} =$	$\frac{9}{10}$ of $1\frac{1}{3} =$
$\frac{5}{6}$ of $7\frac{1}{2} =$	$\frac{5}{9}$ of $3\frac{1}{3} =$	$\frac{7}{6}$ of $8\frac{5}{8} =$	$\frac{7}{4}$ of $1\frac{1}{12} =$
$\frac{3}{7}$ of $9\frac{2}{3} =$	$\frac{7}{12}$ of $7\frac{1}{2} =$	$\frac{9}{4}$ of $2\frac{1}{3} =$	$\frac{8}{3}$ of $1\frac{1}{3} =$
$\frac{4}{9}$ of $5\frac{1}{2} =$	$\frac{3}{11}$ of $1\frac{2}{3} =$	$\frac{10}{3}$ of $3\frac{1}{2} =$	$\frac{7}{5}$ of $6\frac{1}{5} =$
$\frac{7}{8}$ of $12\frac{1}{2} =$	$\frac{8}{5}$ of $4\frac{1}{3} =$	$\frac{5}{2}$ of $7\frac{1}{2} =$	$\frac{4}{3}$ of $5\frac{1}{3} =$

3. *Mental Solutions—without Analysis*

Very many exercises should be given in mental solutions especially in the part-fraction form. The solutions should be brief in form, being shorn of much of the analysis work of the written solutions.

a. In $\frac{3}{4}$ of $5\frac{1}{2} =$, the child should give it, "$\frac{3}{4}$ of 4 is 3, $\frac{3}{4}$ of $\frac{4}{12}$ is $\frac{3}{12} - 3\frac{3}{12}$ or $3\frac{1}{4}$."

b. In $\frac{5}{6}$ of $8\frac{1}{2} =$ the child should give it, "$\frac{5}{6}$ of 6 is 5, $\frac{5}{6}$ of $\frac{30}{12}$ is $\frac{25}{12}$ or $2\frac{1}{12} - 7\frac{1}{12}$.

The teacher will note that the analysis is not used. The pupil states what he is doing with the result of his mental calculations, first the parts of the work, then the final result.

c. $\frac{5}{6}$ of $1\frac{7}{9} =$

"$\frac{5}{6}$ of $\frac{16}{9}$ is $\frac{5}{6}$ of $\frac{48}{27}$ or $1\frac{13}{27}$."

EXERCISES FOR CLASS USE

$\frac{2}{3}$ of $1\frac{3}{8} =$	$\frac{4}{3}$ of $21\frac{1}{2} =$	$\frac{7}{9}$ of $1\frac{1}{4} =$
$\frac{5}{6}$ of $22 =$	$\frac{2}{3}$ of $37\frac{1}{2} =$	$\frac{3}{4}$ of $19 =$
$\frac{3}{4}$ of $23 =$	$\frac{5}{6}$ of $16\frac{1}{2} =$	$\frac{8}{9}$ of $10\frac{1}{2} =$
$\frac{5}{9}$ of $46\frac{1}{2} =$	$\frac{7}{12}$ of $27 =$	$\frac{1}{4}$ of $29\frac{1}{2} =$
$\frac{7}{8}$ of $34 =$	$\frac{5}{8}$ of $25 =$	$\frac{5}{12}$ of $25\frac{1}{4} =$
$\frac{11}{12}$ of $26 =$	$\frac{2}{3}$ of $23\frac{1}{2} =$	$\frac{8}{3}$ of $1\frac{2}{3} =$

SUMMARY.

1. Exercises in which there is but *one* unit to be subdivided.
   - (1) Object work without solutions (without =) for *imagery*. (*A*, *1*.)
   - (2) Object work with solutions (with =) for the *oral* and the *written language* (analytical). (*A*, *2*.)
   - (3) Work without objects with solutions (with =) *for* analysis—
     - (*a*) Written solutions. (*A*, *3*.)
     - (*b*) Mental solutions. (*A*, *4*.)
2. Exercises in which there are *two or more* units to be *subdivided*—the dividends, when fractional, having numerators that are numerically limited. (*B*, *Note 1*.)

   This is to develop that when units are subdivided in partition, the exchange must be for units having the lowest possible denominator. (*B*, *1 a*.)
   - (1) Object work without solutions for imagery. (*B*, *1*.)
   - (2) Object work with solutions for the oral and the written language (analytical). (*B*, *2*.)
   - (3) Work without objects—written (*B*, *3*), and mental (*B*, *4*).

3. Exercises in the partition of miscellaneous *fractions*. This presents the problem of the partition of fractions whose numerators are not limited numerically. (*C*, *1 a*.)
4. Exercises in part-fraction and all-fraction partitions.

   This is to develop the two methods of partition and to teach to choose between them in regular work, using the part-fraction method when practicable. (*D*, *1 a*.)

   (1) Objective work for the development of the terms. (*D*, *1*.)

   (2) Work without objects for double solutions and for training in choice of method. (*D*, *2*.)

   (3) Mental solutions for double solutions or choosing. (*D* 3.)
5. Mixed number exercises.

   These are for training in oral and in written solutions in the two forms to the end that the pupil may develop the power to choose and the habit of choosing the proper method for each problem to be solved. (*E*.)

Time required for the work of this chapter, including reviews of Chapters II–XXII, twenty days. Total time for Chapters II–XXIII, 130 days.

CHAPTER XXIV

MULTIPLICATION OF FRACTIONS

Topics:—The Forms of the Constructions. Multiplication Terms. How to Discriminate Multiplier and Multiplicand. The Form of Multiplication Analysis

A. The Multiplier a Fraction—Constructions in "Horizontal" Form

1. Constructions without Solutions

a. Aim.—This step is primarily to give experience in constructions of dictations in which the multipliers are fractions or mixed numbers. It continues the work of Chapter XXI, completing the construction work of the subject. Incidentally it introduces the term multiplicand, continues the use of the other multiplication expressions—multiply, multiply by, multiplier, and product—and continues the reading of exercises backward and forward. (See Chapter XXI, Introductory Note.)

b. Method of Presentation

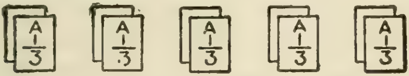
This work is to be in the form of class exercises, each pupil constructing under a leader (first one and then another being selected for this work) who "talks" and constructs, the other members of the class timing their constructions with this leader's oral work.

The dictations as suggested below involve nothing that is new in reading or constructions until the dictation $\frac{1}{2}\times\frac{2}{3}$ is reached. The method of presentation consists of exercises (review) arranged in sequence leading up to this $\frac{1}{2}\times\frac{2}{3}$. The pupil will easily infer the meaning. *Usually* the *form* of the construction will have to be taught. The pupil is repeatedly led up to such forms of construction.

(i) $5\times\frac{2}{3}$

"This means 5 times $\frac{2}{3}$, 5 two-thirds, two-thirds 5 times.

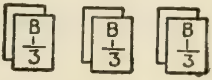
"I make $\frac{2}{3}$ 5 times."



(ii) $3\times\frac{2}{3}$

"This means 3 times $\frac{2}{3}$, 3 two-thirds, $\frac{2}{3}$ 3 times.

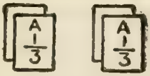
"I make $\frac{2}{3}$ 3 times."



(iii) $2\times\frac{2}{3}$

"This means 2 times $\frac{2}{3}$, twice $\frac{2}{3}$, $\frac{2}{3}$ twice.*

"I make $\frac{2}{3}$ twice."



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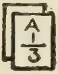
* If the pupil uses the expression "2 times" lead him to say it in another way also. We must bring "twice" and "once" into use. These two words are more common and vivid than "two times" and "one time." "Twice" and "once" materially assist the *inference* when he faces $\frac{1}{2}\times\frac{2}{3}$.

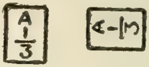
(iv) $1 \times \frac{2}{3}$.

"This means once $\frac{2}{3}$, $\frac{2}{3}$ once. I make $\frac{2}{3}$ once."

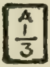


(v) $\frac{1}{2} \times \frac{2}{3}$.

"This means $\frac{1}{2}$ times $\frac{2}{3}$, $\frac{2}{3}$ half a time. I make $\frac{2}{3}$ half a time."



NOTE 1.—It is here that some assistance may be required as to the form of the concrete expression. The pupil will almost certainly show the "*half a time*," but he may have to be shown that in expressing the "half a time" called for, he must show also the *other* "half a time"—the "unseen" half.



This one construction having been given him he will appreciate, in the light of his concrete partition experiences, that the "$\frac{2}{3}$ taken half-a-time" must necessarily have an "*unseen*" feature—that there must be the "half-a-time" called for and the "half-a-time" *not* called for and consequently "hidden."

We sometimes hear it in schools that "of" means *times*. That this is not strictly true becomes apparent in the concrete expression of multiplication. As a matter of fact, construction develops that instead of "of" meaning *times* or ×, the reverse is true—times or ×, when the multiplier is a fraction, has the effect of "of"—partition.

(vi) The same work is now continued using the following exercises in order:

(1) $3 \times \frac{4}{7}$
$1 \times \frac{4}{7}$
$\frac{1}{2} \times \frac{4}{7}$
$3 \times \frac{3}{4}$
$1 \times \frac{3}{4}$
$\frac{2}{2} \times \frac{3}{4}$

(2) $1 \times \frac{6}{7}$
$\frac{2}{3} \times \frac{6}{7}$
$1 \times \frac{10}{3}$
$\frac{3}{5} \times \frac{10}{3}$
$\frac{1}{6} \times \frac{12}{15}$
$\frac{5}{8} \times \frac{8}{9}$

(3) $\frac{3}{7} \times \frac{14}{15}$
$\frac{2}{3} \times 1\frac{1}{2}$
$\frac{3}{4} \times 1\frac{1}{3}$
$\frac{2}{3} \times 1\frac{5}{7}$
$\frac{5}{6} \times \frac{1}{2}$ *
$\frac{5}{8} \times \frac{2}{3}$

(4) $\frac{3}{5} \times \frac{2}{3}$
$\frac{3}{4} \times 1\frac{1}{2}$
$\frac{5}{7} \times 1\frac{1}{6}$
$\frac{6}{7} \times 1\frac{2}{3}$
$\frac{5}{12} \times 2\frac{2}{3}$
$\frac{5}{9} \times 1\frac{1}{5}$

(5) $\frac{3}{5} \times \frac{4}{2}$
$\frac{3}{8} \times 1\frac{1}{4}$
$\frac{7}{9} \times 2\frac{1}{2}$
$\frac{4}{5} \times \frac{7}{8}$
$\frac{5}{6} \times \frac{5}{7}$
$\frac{5}{7} \times \frac{3}{4}$

NOTE 2. *The Terms in Multiplication.*—It is very important that the teacher in connection with this work make the fullest possible use of the terms multiplication, multiply, multiply by, multiplier, multiplicand, and product. The dictations should frequently be given in these terms: multiply $1\frac{3}{5}$ by $2\frac{1}{5}$; what is the product of $2\frac{1}{4}$ multiplied by $1\frac{7}{8}$? the multiplier is $1\frac{1}{2}$, the multiplicand $3\frac{1}{4}$; etc. She may be compelled to go out of her way, almost, to make use of them; but the special effort should be made, if necessary, in order to bring the terms into use by the pupils.

There is much of looseness and inconsistency in the use of the terms multiplier and multiplicand due to the fact that ordinarily they are not developed or taught, but are simply thrown into use without regard to the measuring acts which they express when put to the test of organized objective work. It is with strict reference to meanings determined under such a test that they should come into use.

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* When the leader reaches this dictation he will follow the experiences of partition and "exchange" the $\frac{1}{2}$ for the 6th term of the $\frac{1}{2}$ series—$\frac{6}{11}$. This statement of the "exchange" must enter into his oral work.

The question arises, Which is which in 4×5? Is 4 the multiplicand and 5 the multiplier, or is 5 the multiplicand? Shall we write a given applied problem 4×5 or 5×4? Does it matter which of the two forms we use? To permit now one and now the other form would be to confuse all "times" imagery under any objective number system. Objective work, whatever its basis, if consistent with itself, compels the one or the other form exclusively—compels it in integral and in fractional work wherever "×" is used to express an act of measurement. It should be said, however, that it matters less which way the question is settled than that having settled it we adhere strictly to that form in all class work.

The question to be settled now is, Which of these two interpretations of 4×5 bears the test of harmony with natural untechnical speech when applied to physical acts of measuring?

When the child in his earliest number training was asked, without explanation of the term, to show 3 2's, 3 5's, 2 4's, or other similar concrete expression, he proceeded at once to the construction. It was natural, untechnical language and he easily inferred its meaning. When later he was told that his written 4 5's could be written also 4×5, he saw in this technical form of expression no serious or confusing change. There was nothing out of the natural in reading 4×5 as 4 5's. Later the reading of 4×5 in regular technical language, as 4 times 5, made merely another simple link in the chain. In each of the forms of expression, the second was the concrete term—the multiplicand.

Now 4 5's with the second term as the multiplicand is the only common familiar language in which we can express in number signs four groups of five units each.

If on the other hand we attempt to lead the child from the familiar 4 5's to the other technical form, 5×4, with its oral expression, 5 multiplied by 4, its change of form and its highly technical "multiplied by" will rob it of every semblance of the original 4 5's. The 4 5's and the 5 multiplied by 4 cannot be made parts of the same objective system, there being no inter-connecting link in familiar language or written form.

Any scheme of objective number teaching that would aspire to the dignity of a *system* must provide that the second term of such an expression as 4×5 shall be the multiplicand in all cases.

If the problem for solution relates to six horses at $140 each, the statement must be 6×$140. If it relates to 140 hats at $6 each, it must take the form 140×$6. In a multiplication in vertical form as $\begin{array}{r}361\\ \times\ 24\end{array}$ the multiplicand may be the upper number or the lower number according to convenience. In the horse problem above, we solve it thus $\begin{array}{r}140\\ \times\ \ 6\end{array}$ without regard to which of the factors is concrete. In the hat problem, we use the same form $\begin{array}{r}140\\ \times\ \ 6\end{array}$ the $ not being attached.

All applied problems should be solved in horizontal form. If we are computing the value of 125 boxes at $24 each, our statement should be horizontal 125×$25=. The multiplication of 24×125 should be performed on separate paper or in a separate part of the same paper $\begin{array}{r}125\\ \times\ 24\end{array}$ and the result written in the horizontal statement. Vertical multiplications should be regarded as a means to an end and the factors *not* *marked* as concrete and abstract—as multiplier and multiplicand. (See p. 484 Note.)

2. *Constructions with Solutions—Analysis Form*

a. AIM.—This work has to do with solutions of the class of exercises constructed in *1* above. The constructions as well as the solutions in this step must be in analysis form after the plan of the partition work in Chapter XXIII (*A*, *2 b* (i) and (ii).)

b. CLASS EXERCISES WITH "LEADER AND WRITER"

The "leader's" work will be as follows:

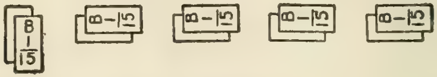
(i) $\frac{3}{5} \times \frac{2}{3} =$

"This means $\frac{3}{5}$ times $\frac{2}{3}$, $\frac{2}{3}$ three fifths of a time.

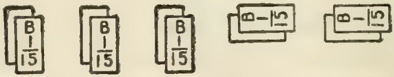
"I must make $\frac{2}{3}$ one fifth of a time, then three-fifths of a time.

"First I exchange $\frac{2}{3}$ for $\frac{10}{15}$, the 5th term of the $\frac{2}{3}$ series.

"I make $\frac{2}{3}$, $\frac{1}{5}$ of a time."



"I make $\frac{2}{3}$, $\frac{3}{5}$ of a time."



"One-fifth times * $\frac{2}{3}$ is $\frac{1}{5}$ times * $\frac{10}{15}$ or $\frac{2}{15}$.

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* After two or three exercises, the teacher will discover that the pupils in *oral* work and later in

Three fifths times $\frac{2}{3}$ is 3 times $\frac{2}{15}$ or $\frac{6}{15}$, I exchange the $\frac{6}{15}$ for $\frac{2}{5}$. I have $\frac{2}{5}$.")

The "writer's" work will be as follows:

$\frac{3}{5}\times\frac{2}{3}=\frac{2}{5}$. (The $\frac{2}{5}$ is written here after the solution.)

$\frac{1}{5}\times\frac{2}{3}=\frac{1}{5}$ of $\frac{10}{15}=\frac{2}{15}$.
$\frac{3}{5}\times\frac{2}{3}=3\times\frac{2}{15}=\frac{6}{15}=\frac{2}{5}$.

(ii) $\frac{3}{4}\times 1\frac{1}{2}=$

The "leader's" oral work is not unlike that given in (i) above for $\frac{3}{5}\times\frac{2}{3}=$. The statements of the meaning, the series, the exchange, and the two constructions and statements for the analysis are in the same form. The "writer's" work is as follows:

$\frac{3}{4}\times 1\frac{1}{2}=1\frac{1}{8}$. (The $1\frac{1}{8}$ is written after the solution.)

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written analysis will want to use "of" instead of "times" or × in this part of an analysis. It should not be discouraged. It is the natural word or sign. There is no objection to a proper interchange. The "writer's" work in this analysis should be accepted if as follows:

$\frac{3}{5}\times\frac{2}{3}=$
$\frac{1}{5}$ of $\frac{2}{3}=\frac{1}{5}$ of $\frac{10}{15}=\frac{2}{15}$
$\frac{3}{5}\times\frac{2}{3}=3\times\frac{2}{15}=\frac{6}{15}=\frac{2}{5}$

The very greatest freedom that is allowable in such an analysis, however, is as follows:

$\frac{3}{5}\times\frac{2}{3}=$
$\frac{1}{5}$ of $\frac{2}{3}=\frac{1}{5}$ of $\frac{10}{15}=\frac{2}{15}$
$\frac{3}{5}$ of $\frac{2}{3}=3\times\frac{2}{15}=\frac{6}{15}=\frac{2}{5}$

$\frac{1}{4}$ of $1\frac{1}{2} = \frac{1}{4}$ of $\frac{3}{2} = \frac{1}{4}$ of $\frac{12}{8} = \frac{3}{8}$.
$\frac{3}{4} \times 1\frac{1}{2} = 3 \times \frac{3}{8} = \frac{9}{8} = 1\frac{1}{8}$.

(iii) Exercises for class use.

$\frac{2}{3} \times \frac{1}{2} =$
$\frac{3}{5} \times \frac{3}{4} =$
$\frac{4}{7} \times \frac{14}{15} =$
$\frac{5}{9} \times \frac{6}{10} =$
$\frac{8}{10} \times \frac{5}{8} =$
$\frac{5}{6} \times \frac{10}{12} =$

$\frac{5}{6} \times \frac{2}{3} =$
$\frac{8}{9} \times \frac{3}{8} =$
$\frac{3}{7} \times 1\frac{1}{2} =$
$\frac{6}{7} \times 2\frac{1}{3} =$
$\frac{9}{8} \times 1\frac{1}{3} =$
$\frac{6}{5} \times 2\frac{1}{2} =$

$\frac{2}{3} \times 2\frac{1}{4} =$
$\frac{3}{6} \times 1\frac{1}{3} =$
$\frac{5}{12} \times 2\frac{2}{5} =$
$\frac{5}{9} \times 1\frac{1}{5} =$
$\frac{7}{10} \times 1\frac{1}{4} =$
$\frac{4}{3} \times 1\frac{1}{4} =$

NOTE 3.—The suggestions in Chapter XXIII C, Note, apply here and to all exercises hereafter in multiplication of fractions *horizontal* form.

3. Seat Work without Objects—Analysis Form

These exercises are written out by the pupils on their papers in analysis form as in *2* above.

EXERCISES FOR CLASS USE

$\frac{3}{8} \times \frac{5}{4} =$
$\frac{5}{7} \times \frac{5}{9} =$
$\frac{3}{10} \times \frac{8}{11} =$
$\frac{6}{7} \times \frac{2}{5} =$
$\frac{5}{12} \times \frac{7}{8} =$
$\frac{7}{9} \times \frac{4}{5} =$
$\frac{11}{12} \times \frac{3}{8} =$
$\frac{8}{5} \times \frac{6}{7} =$
$\frac{9}{12} \times \frac{5}{4} =$
$\frac{8}{10} \times \frac{6}{12} =$

$\frac{8}{7} \times 3\frac{1}{2} =$
$\frac{4}{3} \times 2\frac{1}{4} =$
$\frac{3}{5} \times 2\frac{3}{5} =$
$\frac{7}{8} \times 1\frac{1}{5} =$
$\frac{7}{3} \times \frac{7}{8} =$
$\frac{5}{4} \times \frac{9}{11} =$
$\frac{7}{9} \times \frac{5}{3} =$
$\frac{9}{5} \times \frac{8}{3} =$
$\frac{3}{4} \times \frac{5}{7} =$
$\frac{5}{8} \times \frac{7}{3} =$

$\frac{5}{12} \times 8\frac{1}{2} =$
$\frac{5}{9} \times 6\frac{2}{3} =$
$\frac{6}{7} \times 1\frac{2}{3} =$
$\frac{7}{9} \times 2\frac{2}{3} =$
$\frac{5}{6} \times \frac{7}{8} =$
$\frac{8}{9} \times \frac{10}{12} =$
$\frac{4}{2} \times \frac{5}{7} =$
$\frac{5}{3} \times \frac{5}{4} =$
$\frac{6}{5} \times \frac{6}{10} =$
$\frac{8}{3} \times \frac{7}{2} =$

$\frac{4}{3}\times5\frac{1}{2}=$

$\frac{6}{5}\times3\frac{1}{4}=$

$\frac{10}{8}\times2\frac{3}{4}=$

$\frac{7}{5}\times3\frac{1}{3}=$

$\frac{3}{5}\times\frac{6}{4}=$

$\frac{9}{8}\times\frac{4}{3}=$

$\frac{12}{3}\times1\frac{5}{7}=$

$\frac{11}{8}\times\frac{10}{4}=$

$\frac{12}{10}\times\frac{8}{5}=$

$\frac{11}{3}\times\frac{4}{2}=$

$\frac{7}{8}\times\frac{14}{9}=$

$\frac{5}{12}\times\frac{18}{11}=$

$\frac{5}{11}\times\frac{4}{9}=$

$\frac{5}{9}\times\frac{12}{7}=$

$\frac{5}{3}\times\frac{12}{8}=$

$\frac{20}{8}\times\frac{12}{5}=$

$\frac{21}{9}\times\frac{5}{4}=$

$\frac{18}{5}\times1\frac{3}{12}=$

$\frac{16}{3}\times\frac{4}{2}=$

$\frac{15}{7}\times\frac{2}{5}=$

4. Mental Solutions—without Analysis

AIM.—This should not be purely mental work but seat written work with mental computations. In practical life the work that must be *mental* is partition for the most part, not multiplication, hence the emphasis on mental *partition* rather than *multiplication.*

The exercises in this step should be simple, should not be given until after very much experience under *3* above, and should be written in the briefest possible form.

FORM OF WRITTEN WORK

$\frac{3}{4}\times\frac{5}{6}=\frac{3}{4}$ of $\frac{20}{24}=\frac{15}{24}=\frac{5}{8}$.

In this solution, no questions are to be asked as to mental processes. When the child does objective work, and when he writes out his solutions in analysis form, the mental processes are fully evident. In this written-mental work the teacher should be satisfied with correct statements.

$\frac{5}{8}\times 1\frac{1}{2}=\frac{5}{8}\times\frac{3}{2}=\frac{5}{8}$ of $\frac{24}{16}=\frac{15}{16}$.

EXERCISES FOR CLASS USE

$\frac{2}{3}\times\frac{5}{6}=$

$\frac{4}{5}\times\frac{3}{4}=$

$\frac{6}{9}\times 2\frac{1}{2}=$

$\frac{4}{2}\times\frac{5}{6}=$

$\frac{5}{8}\times 1\frac{1}{3}=$

$\frac{2}{3}\times 4\frac{1}{2}=$

$\frac{3}{4}\times 2\frac{3}{4}=$

$\frac{9}{8}\times 1\frac{3}{8}=$

$\frac{12}{5}\times 1\frac{3}{4}=$

$\frac{2}{3}\times\frac{9}{10}=$

$\frac{5}{7}\times 1\frac{4}{10}=$

$\frac{7}{8}\times 3\frac{3}{5}=$

$\frac{4}{11}\times 3\frac{1}{7}=$

$\frac{8}{5}\times\frac{3}{7}=$

$\frac{9}{4}\times\frac{3}{4}=$

$\frac{7}{3}\times\frac{2}{3}=$

$\frac{12}{11}\times\frac{3}{2}=$

$\frac{14}{2}\times\frac{3}{11}=$

$\frac{16}{3}\times\frac{6}{7}=$

$\frac{12}{8}\times\frac{4}{5}=$

B. The Multiplier a Mixed Number—Horizontal Solutions

1. Constructions with "Leader" and "Writer" —Analysis Form

a. "HORIZONTAL" EXPLAINED

The term "horizontal" solution is used to designate the all-fraction form of multiplying when one or both of the factors are mixed numbers. When we multiply $1\frac{1}{2}\times 1\frac{1}{3}$ (for illustration) by reducing both of them to improper fractions and then dealing with the problem as we deal with the multiplication of any two fractions, we call it the *horizontal* form. *All* multiplication of fractional (not mixed) numbers is horizontal.

When mixed numbers are multiplied by arranging them vertically as $\begin{array}{r} 1\frac{1}{2} \\ \times 2\frac{1}{3} \\ \hline \end{array}$ and per-forming the operations with them in that form (see *C* below) we designate it as the vertical form.

AIM.—To extend the construction and oral work in "horizontal" solutions to exercises with mixed number multipliers. The work in *A* had to do with exercises having fractional or mixed number multiplicands. This step has to do with exercises in which both factors are mixed numbers.

b. CLASS EXERCISES WITH "LEADER" AND "WRITER"

The teacher chooses a "leader" and a "writer."

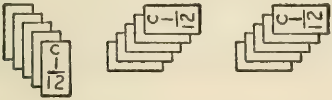
THE "LEADER'S" WORK

(i) $\frac{2}{3}$ * $\times 1\frac{1}{4} =$

"This means $\frac{2}{3}$ times $1\frac{1}{4}$,' $1\frac{1}{4}$ taken $\frac{2}{3}$ of a time.

"$\frac{2}{3}$ times $1\frac{1}{4}$ is $\frac{2}{3}$ times $\frac{5}{4}$.

"First, I make $\frac{1}{3}$ of $\frac{5}{4}$ or $\frac{1}{3}$ of $\frac{15}{12}$.



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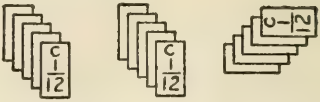
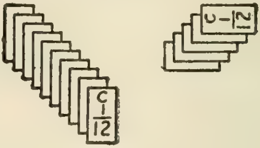
[*] This is not representative of the class of exercises with which the step deals, the multiplier not being a mixed number. It is simply introductory to (ii) below, the construction and oral work of which is exactly the same in form.

"I now make $\frac{2}{3}$ of $\frac{5}{4}$.

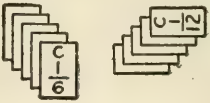


"One third of $\frac{5}{4}$ is $\frac{5}{12}$.

"$\frac{2}{3}$ of $\frac{5}{4}$ is 2 times $\frac{5}{12}$ or $\frac{10}{12}$.



"I exchange the $\frac{10}{12}$ for $\frac{5}{6}$. I have $\frac{5}{6}$."



THE "WRITER'S" WORK

When the leader makes his construction (1) of $\frac{1}{3}$ of $\frac{5}{4}$ and (2) of $\frac{2}{3}$ of $\frac{5}{4}$ the "writer" makes the blackboard diagrams (see Chapter XXII (*B*, Note.)

The "writer's" written work will be—

$\frac{2}{3} \times 1\frac{1}{4} =$

$\frac{1}{3} \times 1\frac{1}{4} = \frac{1}{3} \times \frac{5}{4} = \frac{1}{3}$ of $\frac{15}{12} = \frac{5}{12}$.

$\frac{2}{3} \times 1\frac{1}{4} = 2 \times \frac{5}{12} = \frac{10}{12} = \frac{5}{6}$.

(ii) $1\frac{1}{3} \times 1\frac{1}{5} =$

The teacher selects the "leader" and the "writer."

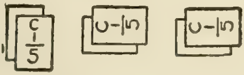
THE "LEADER'S" WORK

"This means $1\frac{1}{3}$ times $1\frac{1}{5}$, $1\frac{1}{5}$ taken $1\frac{1}{3}$ times.

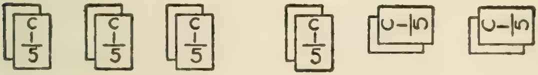
"$1\frac{1}{3}$ times $1\frac{1}{5}$ is $1\frac{1}{3}$ times $\frac{6}{5}$.

NOTE 1.—This is the form in which the leader and the writer will give this statement. The reduction of the multiplier to the improper fraction form will come two or three exercises later as the pupils find by experience that such reduction is an analysis necessity. The leader finds no difficulty, there is no hesitation, in making his *first* analysis statement "$\frac{1}{3}$ times $\frac{6}{5}$ is, etc."; but when he comes to make his *second* analysis statement, he finds that $1\frac{1}{3}$ is not usable—he must change its form to *four-thirds* ($\frac{4}{3}$ times $\frac{6}{5}$ is, etc.). He finds that he cannot make the proper statement without such change. The second exercise and perhaps the third may be given by him in the same mixed number form, but after the two or three experiences he will state such an exercise properly—"'$1\frac{1}{3}$ times $1\frac{1}{5}$ is $\frac{4}{3}$ times $\frac{6}{5}$."

"'First, I make $\frac{1}{3}$ of $\frac{6}{5}$.

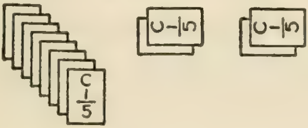


"I now make $\frac{4}{3}$ of $\frac{6}{5}$.

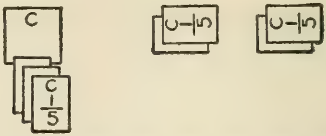


"One-third of $\frac{6}{5}$ is $\frac{2}{5}$.

Four-thirds of $\frac{6}{5}$ is 4 times $\frac{2}{5}$ or $\frac{8}{5}$.



"I exchange five fifths for one. I have $1\frac{3}{5}$."



NOTE 2.—This is what he says finally after a struggle, mental and with his objects.

THE "WRITER'S" WORK

$$1\tfrac{1}{3} \times 1\tfrac{1}{5} = \tfrac{4}{3} \times \tfrac{6}{5} =$$

$$\tfrac{1}{3} \times 1\tfrac{1}{5} = \tfrac{1}{3} \times \tfrac{6}{5} = \tfrac{2}{5}.$$

$$\tfrac{4}{3} \times 1\tfrac{1}{5} = 4 \times \tfrac{2}{5} = \tfrac{8}{5} = 1\tfrac{3}{5}.$$

2. *Seat Work without Objects—Analysis Form*

In this, the pupils write the exercises on their papers from written dictations using the forms of statements and the arrangement as in "The Writer's Work," in *1*, *b* (ii) above.

EXERCISES FOR CLASS USE

$2\frac{1}{3} \times 2\frac{1}{2} =$
$1\frac{3}{4} \times 3\frac{1}{3} =$
$2\frac{1}{4} \times 1\frac{2}{3} =$
$3\frac{2}{3} \times 1\frac{1}{8} =$
$3\frac{3}{4} \times 1\frac{2}{3} =$

$4\frac{1}{5} \times 2\frac{2}{3} =$
$1\frac{4}{5} \times 1\frac{3}{5} =$
$1\frac{5}{6} \times 2\frac{1}{7} =$
$2\frac{1}{8} \times 1\frac{1}{2} =$
$1\frac{7}{8} \times 2\frac{2}{3} =$

$1\frac{1}{9} \times 2\frac{1}{7} =$
$3\frac{1}{2} \times 1\frac{4}{5} =$
$1\frac{2}{3} \times 2\frac{2}{3} =$
$1\frac{3}{5} \times 1\frac{3}{4} =$
$2\frac{1}{2} \times 2\frac{1}{2} =$

$4\frac{2}{5} \times 1\frac{1}{2} =$
$1\frac{1}{2} \times 3\frac{2}{5} =$
$2\frac{1}{3} \times 2\frac{1}{4} =$
$1\frac{6}{7} \times 1\frac{1}{3} =$
$2\frac{1}{8} \times 1\frac{1}{2} =$

C. Both Factors Mixed Numbers—Vertical Solutions

This form of solution was fully developed, as to arrangement of the objects in the concrete work, in Chapter XX. Objective work in this step is, therefore, hardly necessary except for now and then an exercise, not too complicated, to keep the imagery fresh.

The pupil will use this form of multiplication (vertical) later on in practical work in preference to the horizontal. It is more simple and more rapid.

a. AIM.—The aim is to teach the pupil to perform the operation indicated in $\begin{array}{r} 3\frac{1}{4} \\ \times 2\frac{2}{3} \\ \hline \end{array}$ in just the same way as he would proceed in multiplying 52 × 43.

In the latter exercise, the multiplier and multiplicand are arranged vertically $\begin{array}{r} 52 \\ \times 43 \\ \hline \end{array}$. The operation is—three times 2, then 3 times 5, then 4 times 2, and then another 4 times 5.

In the vertical multiplication of $3\frac{1}{4}$ by $2\frac{2}{3}$, the operation takes exactly the same form.

$\begin{array}{r} 3\frac{1}{4} \\ \times 2\frac{2}{3} \\ \hline \end{array}$. The order is—$\frac{2}{3}$ times $\frac{1}{4}$, then $\frac{2}{3}$ times 3, then 2 times $\frac{1}{4}$, and then 2 times 3. No deviation from this order of multiplying is permitted.

b. METHOD OF PRESENTATION.

The method of introduction is simply to send a pupil to the blackboard with an exercise, like the 52 times 43 above, to "talk" and write—pointing to each number as he speaks it.

(i) $\begin{array}{r}52\\ \times 43\\ \hline\end{array}$ When he says, "3 times 2," his finger touches the 3 and then the 2. As he says, "3 times 5," his finger goes to the 3 and then to the 5. As he says, "4 times 2," his finger touches the figures in order. As he says, "4 times 5," he touches the 4 and then the 5. Then he adds the partial products.

"Show me again with your finger without writing how you multiplied."

("I said, '3 times 2, then 3 times 5, then 4 times 2, and then 4 times 5'.")

"How would you multiply in this?"— $\begin{array}{r}36\\ \times 81\\ \hline\end{array}$

("I would say, 'Once 6, then once 3, then 8 times 6, then 8 times 3'.")

(ii) "I want you to multiply this"— $\begin{array}{r}3\frac{1}{4}\\ \times 2\frac{2}{3}\\ \hline\end{array}$ "in exactly the same way."

("Two-thirds times $\frac{1}{4}$ is $\frac{2}{12}$ or $\frac{1}{6}$.

"Two-thirds times 3 is 2.

"Two times $\frac{1}{4}$ is $\frac{2}{4}$ or $\frac{1}{2}$.

"Two times 3 is 6.")

The pupil then adds the partial products.

The pupil's written work—the partial products—as he talks *must* be arranged as follows:

$$\begin{array}{rl} 3\frac{1}{4} & \\ \times 2\frac{2}{3} & \\ \hline \frac{1}{6} & (1) \\ 2 & (2) \\ \frac{1}{2} & (3) \\ 6 & (4) \\ \hline 8\frac{2}{3} & \end{array}$$

Another exercise is given for some pupil to go to the blackboard to give the oral and written work.

$$\text{(iii)}\quad \begin{array}{rl} 4\frac{1}{2} & \\ \times 2\frac{5}{8} & \\ \hline \frac{5}{16} & (1) \\ 2\frac{1}{2} & (2) \\ 1 & (3) \\ 8 & (4) \\ \hline 11\frac{13}{16} & \end{array} \qquad\qquad \text{(iv)}\quad \begin{array}{rl} 2\frac{3}{4} & \\ \times 4\frac{7}{8} & \\ \hline \frac{21}{32} & (1) \\ 1\frac{3}{4} & (2) \\ 3 & (3) \\ 8 & (4) \\ \hline 13\frac{13}{32} & \end{array}$$

Three or four such exercises are given before the seat written work is taken up.

c. Exercises for Class Use

$2\frac{1}{2} \times 4\frac{2}{3} =$ (v) * Multiply $5\frac{1}{4}$ by $2\frac{5}{8}$ (v)

---

* The form that the pupil is to use in multiplying (vertical or horizontal) may be indicated by V and H.

$5\frac{3}{4}\times 2\frac{5}{6}=$ (v)
$2\frac{2}{7}\times 3\frac{2}{3}=$ (v)
$7\frac{2}{3}\times 5\frac{4}{5}=$ (v)
$6\frac{4}{5}\times 2\frac{1}{4}=$ (v)
$4\frac{5}{8}\times 2\frac{5}{6}=$ (v)
$8\frac{3}{8}\times 3\frac{1}{6}=$ (v)
$4\frac{5}{9}\times 3\frac{5}{8}=$ (v)

Multiply $4\frac{4}{5}\times 2\frac{5}{9}$ (v)
Find the product of $2\frac{5}{7}$ multiplied by $3\frac{5}{8}$ (v)
The product of $2\frac{1}{8}$ multiplied by $3\frac{1}{4}$ (h)
$1\frac{4}{7}\times 2\frac{2}{5}=$ (h)
$1\frac{7}{8}\times 1\frac{1}{5}=$ (h)
$8\frac{4}{5}\times 3\frac{5}{6}=$ (v)
$9\frac{1}{9}\times 2\frac{5}{8}=$ (v)
$3\frac{1}{3}\times 3\frac{1}{2}=$ (h)
$10\frac{3}{5}\times 9\frac{1}{3}=$ (v)

SUMMARY.

1. Exercises to develop the form of the objective expression where the multiplier is a *fraction*.
    (1) Object work.
        (*a*) Constructions without solutions, for imagery. (*A, 1.*)
        (*b*) Comparison of "of" and "times" as to meaning. (*A, Note 1.*)
        (*c*) The terms multiplicand, multiplier, product, etc. introduced and how to discriminate multiplicand and multiplier. (*A, Note 2.*)
        (*d*) Constructions with solutions.
        These show construction forms, the oral work, and the analysis. (*A, 2.*)

  - (2) Work without objects. (*A*, *3*.)
  - (3) Mental solutions—without analysis. (*A*, *4*.)

2. Exercises to develop the objective forms, the oral work, and the analysis in solutions having mixed number multipliers—
  - (1) Horizontal solutions. (*B*, *1* and *2*.)
  - (2) Vertical solutions. (*C*.)

Time required for the work of this chapter, including reviews of Chapters II–XXIII, fifteen days. Total time for Chapters II–XXIV, 145 days.

CHAPTER XXV

DIVISION

Topic:—Quotients with Complex Fractions and How to Avoid Them in Division Processes

A. Constructions—Complex and Simple Groups

1 Aim.—The aim of this step is to develop the terms complex and simple as applied to fractional groups in uneven division.

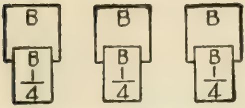
The child understands division construction and terms from Chapter XI. There remains (i) the constructions in which dividend or divisor or both must be reduced to a common denominator before the construction is possible, and (ii) experiences in making such changes in dividend and divisor as are necessary to avoid complex fractional groups where the divisor is uneven.

The aim of this step is to give the pupil the opportunity to devise means of avoiding complex fractional groups rather than to *give* him the information.

2 Method of Presentation

(*a*) Complex and Simple Developed.

(i) The teacher writes $3\frac{3}{4} \div 1\frac{1}{4} =$ on the blackboard and asks the pupil to construct it. The constructions on the desks will be—



A pupil is asked to write the full expression on the blackboard. He writes $3\frac{3}{4} \div 1\frac{1}{4} = 3$.

"How many kinds of units in this construction?"

("Two.")

NOTE 1.—This question is introduced here in order to teach the pupil what the question, which is to be in constant use, refers to. A division construction may be made up entirely of integral units as in $9 \div 4 =$, of fractional units as in $\frac{7}{3} \div \frac{2}{3} =$, or of both integral and fractional units as in this construction. The pupils must be told what, in this respect, the question means. In this case, the construction has both integral and fractional units—two kinds.

"Is there a fractional group?"

("No, there are three full groups.")

(ii) $\frac{8}{4} \div \frac{3}{4} =$

The pupils are, as before, asked to construct; and a pupil writes the full expression, $\frac{8}{4} \div \frac{3}{4} = 2\frac{2}{3}$, on the blackboard.

"How many kinds of units?"*

---

* This question always relates to the *construction* and not to the *dictation*. It means, "How many kinds of units in the concrete expression." In answering this question here or in any other construction the teacher *must insist* on the pupil stating also *which* kind, *fractional* units or *integral* units.

("One kind, fractional units.")

"Is there a fractional group?"

("Yes, two-thirds.")

"What does the numerator tell us?"

("It tells us how many are left over toward another group.")

"What does the denominator tell?"

("It tells us how many in a full group.")

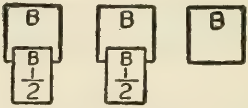
(iii) $6 \div 4 =$

This, as before, is constructed on the desks, the full expression written on the blackboard, and the four questions suggested in ii asked.

(iv) $4 \div 1\frac{1}{2} =$

The constructions, the written expression on the blackboard, and the questions as before.

The constructions will be—



There will be some hesitation about the *written* expression. The pupils, however, must be left to decide for themselves how to express it. They will decide to write the expression $4 \div 1\frac{1}{2} = 2\frac{1}{1\frac{1}{2}}$*

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* The *only* part, if any, permissible for the teacher to take in this construction or in the written fraction, if pupils hesitate, would be to ask the questions—

"Is there a fractional group?"

"How many kinds of units?"

("Two.")

"Is there a fractional unit?"

("Yes.")

"What does the numerator tell?"

("It tells how many left over to make another group.")

"What does the denominator tell?"

("It tells how many in a group.")

"The fraction is right, but it is a *mixed up* fraction. We have them frequently. We call this a *complex* fraction."

(The term is written on the blackboard—complex fraction.)

"The fractions that are *not* complex are called simple fractions."

(This term simple fraction is written on the blackboard beside the other term—

$$\left\{\begin{array}{l}\text{complex fraction}\\ \text{simple fraction}\end{array}\right\}$$

NOTE 2.—The construction work of $4 \div 1\frac{1}{2} =$ must be left on each desk—pushed back so that another and then another construction may be put with it for reference later. The plan is to have several construc-

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"What does the numerator tell?"

"What does the denominator tell?"

It is not ordinarily necessary for the teacher to take *any* part—ask any questions—in advance of writing ft. The teacher should *wait*.

tions made, the written expression put on the blackboard (these also to be preserved in order) and the questions asked upon each as before—some of these constructions to have simple fractions, others to have complex fractions. After these several exercises have been constructed, written, discussed by the questions, and laid aside, the constructions are to be reviewed one after the other in order to let the pupils discover the fact that fractional groups are *complex* when and only *when the construction has two kinds* of units. To arrive at this very important conclusion we must accumulate many concrete expressions with their written expressions in order that we may examine them one after another in quick succession.

The teacher will notice by this that the *very* important question of the four is that which asks as to the *number of kinds* of units in the concrete expression. It would be wise to ask that as the *first* and then as the *fifth* question with each construction. In other words, ask it twice each time.

(v) $6 \div 2\frac{1}{2} =$

The constructions are made, the question is asked, the full expression written on the blackboard, the statement made by the pupils that the fractional group is a complex fraction, the concrete expression laid aside (see Note A below), and the written expression preserved as before.

(vi) $\frac{6}{3} \div \frac{4}{3} =$

This is dealt with in the same way. The fractional group will be found to be a *simple* fraction.

(*b*) Exercises for Construction Experience:

$4\frac{1}{2} \div 3 =$ $3\frac{1}{2} \div 1\frac{1}{2} =$
$6\frac{1}{2} \div 2\frac{1}{2} =$ $4\frac{1}{2} \div 1\frac{1}{4} =$
$\frac{5}{4} \div \frac{3}{8} =$* $\frac{7}{3} \div \frac{5}{6} =$*
$7 \div 5 =$ $7\frac{1}{2} \div 3\frac{1}{4}$

These exercises are used in the same way one after the other and the concrete work is set aside for further examination. (See Note A, below).

(*c*) Constructions Re-examined.

The teacher takes up with the class the re-examination of the constructions that have been laid aside on the desks and the written solutions for them which remain on the black-board. (See Note *2;* and Note A, below.)

"Let us look at our first construction—four divided by one and one half."

"How many kinds of units in your construction?"

("Two, fractional and integral.")

"Is there a fractional group?"

("Yes.")

"What does the numerator tell?"

("It tells how many left over for another group.")

---

* In a construction of this kind the pupil will not hesitate although construction is not possible except by exchanging the dividend so that dividend and divisor have a common denominator.

"What does the denominator tell?"

("It tells how many in a group.")

"Is the fractional group simple or complex?"

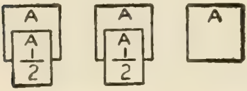
("Complex.")

"How many kinds of units in the construction?"

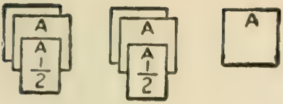
("Two.")

The teacher now takes up another and then another construction in the same way asking the first question (as to the kinds of units in the construction) the second time in each case. This brings the question as to whether the fractional group is simple or complex and the question as to the kinds of units in the construction close together. In this way the pupils will eventually note the connection—when the construction has *two kinds* of units, the fractional group, if any, is *complex*: and when the construction has but *one kind*, the fractional group is *simple*.

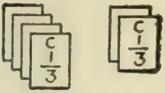
Note A.—This shows the constructions that the pupils have made and put aside on their desks for examination later as explained above in Note 2. It shows also the corresponding full written expressions which were left on the blackboard for examination with the constructions as suggested in the same note.



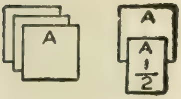
$6 \div 2\frac{1}{2} = 2\frac{1}{2\frac{1}{2}}$



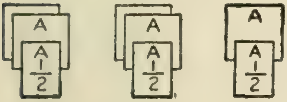
$4 \div 1\frac{1}{2} = 2\frac{1}{1\frac{1}{2}}$



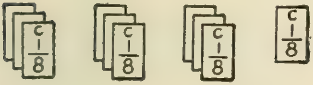
$\frac{6}{3} \div \frac{4}{3} = 1\frac{2}{4} = 1\frac{1}{2}$



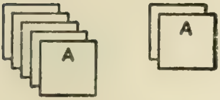
$4\frac{1}{2} \div 3 = 1\frac{1\frac{1}{2}}{3}$



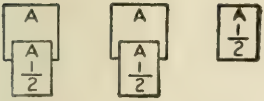
$6\frac{1}{2} \div 2\frac{1}{2} = 2\frac{1\frac{1}{2}}{2\frac{1}{2}}$



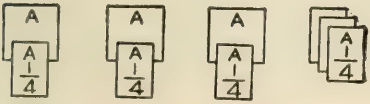
$\frac{5}{4} \div \frac{3}{8} = \frac{10}{8} \div \frac{3}{8} = 3\frac{1}{8}$



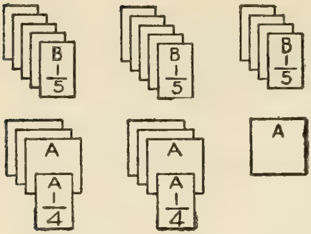
$7 \div 5 = 1\frac{2}{5}$



$3\frac{1}{2} \div 1\frac{1}{2} = 2\frac{\frac{1}{2}}{1\frac{1}{2}}$



$4\frac{1}{2} \div 1\frac{1}{4} = 3\frac{\frac{3}{4}}{1\frac{1}{4}}$



$\frac{7}{3} \div \frac{5}{6} = \frac{14}{6} \div \frac{5}{8} = 2\frac{4}{5}$

*

$7\frac{1}{2} \div 3\frac{1}{4} = 7\frac{2}{4} \div 3\frac{1}{4} = 2\frac{1}{3\frac{1}{4}}$

(*d*) THE RESULTS SUMMARIZED.—The review of the constructions being completed, the following questions designed to effect a summarizing by the pupil of the results of this work are asked one after the other with a view to organizing these results into principles which shall govern future work in division.

"When is the fractional group always complex?"

("When the groups have two kinds of units.")

"When is the fractional group always simple?"

("When the groups have only one kind of units.")

"When we have two kinds of units in the groups, what of the fractional group, if any?"

("The fractional group will be complex.")

"When we have only one kind of unit in the

(* If the pupil should construct so as to leave the part-group as 4ths and give the written statement $7\frac{1}{2} \div 3\frac{1}{4} = 7\frac{2}{4} \div 3\frac{1}{4} = 2\frac{\frac{4}{4}}{3\frac{1}{4}}$ it should be accepted as perfectly satisfactory.)

groups, what of the fractional group, if any?"

("The fractional group will be simple.")

"If we do not want the fractional group, if any, to be complex, what may be done?"

("We must put only one kind of unit into the group.")

As this matter of simple and complex fractional groups is the important feature of division development, these questions, applied to the constructions in Note A and to other constructions of the same kind, should be put to several if not to all members of the class before passing to the next step. The questions should be asked again to-morrow and the next day and the next. The aim is to make the pupil self-dependent in future division work, acting upon the principles which the questions are designed to establish.

(3) COMPLEX FRACTION QUOTIENTS

(*a*) The rule for complex groups.

In this step of the presentation the pupils are told that people do not want the fractional groups in a division exercise to be complex. People want us to construct, if possible, so that the groups shall *not* be complex. This prohibition of complex fractional groups may be put into a *rule* form—"*All quotient fractions must be simple.*"

(*b*) Constructions to avoid complex groups.

The teacher now takes up again the constructions on the pupils' desks (see Note A) with a view to give each child an opportunity to test his ingenuity in so reconstructing the concrete work, when necessary, that all fractional groups will be simple.

This is done by calling the attention of each pupil to his first construction (that of $4 \div 1\frac{1}{2} =$); asking again the questions suggested in *a* (ii); asking for the rule about complex fractional groups ("All quotient fractions must be simple"); and then, "Can you fix this construction so that the fractional group will be simple?" Each child should labor upon his re-construction until he succeeds in making it conform to the requirement that the groups have only one kind of unit. This unit in this case is, of course, a half. If a pupil by chance exchanges for a unit smaller than a half, it must not be accepted. "You could have used a lower unit." Pupils will soon learn that they must use the lowest possible unit.

After the reconstruction, the written statement must be changed to agree with the new construction.

($4 \div 1\frac{1}{2} = \frac{8}{2} \div \frac{3}{2} = 2\frac{2}{3}$.)

The questions and the rule may be repeated with the second construction ($6 \div 2\frac{1}{2} =$).

The reconstruction must be with halves, the new statement after the change being $6 \div 2\frac{1}{2} = \frac{12}{2} \div \frac{5}{2} = 2\frac{2}{5}$.

With the third, the question is, "Is the fractional group complex or simple?" and, "Do we have to change it?" or, "Is it proper as it is?"

With the fourth and those following, there will be no need of questions, probably. The pupils each for himself will re-construct, where necessary, and rewrite the full written statements of the re-constructed expressions.

(*c*) Seat work in avoiding complex groups.

The exercises given here are for additional experience in constructing so as to avoid complex fractional groups. The pupil should be left to make the constructions in his own way. Some pupils, for illustration, may construct the expression without foreseeing that there will be a complex fractional group, and finding one may break up the construction and begin all over again. Suggestions are out of order in such cases. The pupil should be left to work out the problems in his own way provided that the fractional unit with which he forms his groups finally are the lowest that can be used. By leaving him to work out his own problems here, we give him an opportunity to discover that there *are* problems.

The pupil constructs and from the concrete expression writes on his paper the full division statement.

$\frac{6}{5} \div \frac{3}{10} =$

$5\frac{1}{8} \div 1\frac{1}{4} =$

$3\frac{1}{4} \div 1\frac{1}{5} =$

$4\frac{2}{3} \div 1\frac{5}{6} =$

$\frac{5}{6} \div \frac{11}{12} =$

$\frac{8}{3} \div \frac{5}{6} =$

$\frac{9}{4} \div \frac{5}{8} =$

$1\frac{1}{4} \div \frac{2}{3} =$

$\frac{8}{4} \div \frac{5}{6} =$

$\frac{2}{3} \div \frac{5}{6} =$

$\frac{7}{3} \div \frac{5}{4} =$[*]

$\frac{8}{6} \div \frac{3}{4} =$

$\frac{12}{9} \div \frac{1}{4} =$

$\frac{10}{8} \div \frac{5}{6} =$

$\frac{5}{8} \div \frac{3}{4} =$

$2\frac{1}{4} \div 1\frac{1}{2} =$

$4\frac{1}{3} \div 1\frac{1}{6} =$

$\frac{6}{8} \div \frac{1}{2} =$

$2\frac{2}{3} \div 1\frac{1}{6} =$

$1\frac{1}{4} \div 1\frac{1}{2} =$

$6\frac{1}{4} \div 3\frac{2}{3} =$

$2\frac{1}{8} \div 1\frac{1}{3} =$

$3\frac{1}{6} \div 2\frac{1}{4} =$

$2\frac{1}{5} \div 1\frac{1}{3} =$

$3\frac{3}{4} \div 4\frac{2}{3} =$

4. Oral Class Exercises to Anticipate the Kind of Fractional Group

a. AIM.—The following exercises are to be written on the blackboard and discussed one by one as to whether the construction, if made, would or would not have a complex fractional group, if any.

The questions before the pupil should not be whether there would or would not be a fractional group, if construction were made; but, Would the fraction, if there should be one, be simple or complex?

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[*] In a case of this kind in which both dividend and divisor are improper fractions, a complex fractional group is possible only by changing the quantities back to mixed numbers before constructing the groups. Constructions with numbers in fractional form always yield simple fractional groups, if any.

The pupil must decide by looking at the written dictation. When he decides that the fractional group, if any, would be simple or would be complex, he must state why he makes such decision.

b. PRESENTATION

$8\frac{1}{2} \div 2\frac{1}{2} =$

"If we were to construct this and there should be a fractional group, would it be simple or complex?"

("It would be complex because each full group would have two integral units and one fractional unit.")

"How can you tell?"

("The problem tells us to take $8\frac{1}{2}$ and make it into groups of $2\frac{1}{2}$.")

$\frac{17}{6} \div \frac{5}{4} =$

"If we were to construct this, would the fractional group, if any, be simple or complex?"

("It would be simple because each full group would have all fractional units.")

"How do you know this?"

("This problem tells us to take $\frac{17}{6}$ and make it into groups of $\frac{5}{4}$.")

The above exercises show the general plan of class work for the exercises given in *c* below.

c. EXERCISES FOR CLASS USE.

$9\frac{3}{4} \div 5\frac{1}{2} =$	$\frac{9}{4} \div \frac{8}{16} =$	$\frac{5}{8} \div \frac{2}{3} =$	$7\frac{1}{7} \div 2\frac{1}{3} =$
$6\frac{3}{5} \div 2\frac{1}{3} =$	$\frac{10}{2} \div \frac{6}{5} =$	$7\frac{8}{9} \div 2\frac{3}{6} =$	$2\frac{4}{5} \div \frac{8}{2} =$
$7\frac{5}{8} \div 2\frac{2}{3} =$	$3\frac{3}{8} \div 2\frac{2}{3} =$	$9\frac{3}{4} \div 2\frac{4}{5} =$	$\frac{34}{24} \div \frac{1}{8} =$
$5\frac{1}{2} \div 3\frac{3}{5} =$	$2\frac{11}{12} \div 1\frac{1}{8} =$	$\frac{4}{7} \div \frac{3}{2} =$	$\frac{28}{12} \div \frac{9}{8} =$
$12\frac{2}{3} \div 3\frac{3}{4} =$	$3\frac{1}{4} \div 3\frac{1}{3} =$	$\frac{15}{4} \div \frac{5}{3} =$	$14\frac{1}{2} \div 2\frac{1}{7} =$

B. Seat Work without Objects

The pupil has now had enough construction experience to enable him to work intelligently without objects. He has not been told how to solve his problems. He has been left to his own resources to construct, his exercises being held to two conditions—

(1) That his fractional groups must be *simple*, and

(2) That in construction with fractional units he must use those having the lowest denominator possible.

In his seat-work without objects he is to make solutions in his own way subject only to these two conditions.

We shall find him reducing his dividend and divisor to like units (fractional) and dividing (making his groups) with fractional units. This is as it should be.

EXERCISES FOR CLASS USE

$8\frac{1}{3} \div 3\frac{3}{5} =$
$9\frac{4}{7} \div 9\frac{2}{3} =$
$6\frac{4}{9} \div 3\frac{3}{4} =$
$12\frac{4}{11} \div 3\frac{1}{3} =$
$10\frac{5}{8} \div 3\frac{5}{12} =$
$8\frac{6}{7} \div 6\frac{3}{4} =$
$9\frac{1}{3} \div 2\frac{4}{7} =$

$\frac{15}{4} \div \frac{7}{5} =$
$\frac{12}{7} \div 1\frac{3}{4} =$
$2\frac{3}{8} \div \frac{5}{12} =$
$7\frac{1}{7} \div 4\frac{1}{4} =$
$\frac{15}{3} \div 1\frac{7}{8} =$
$\frac{14}{5} \div 1\frac{4}{7} =$
$2\frac{3}{4} \div \frac{7}{6} =$
$12\frac{2}{3} \div 3\frac{4}{5} =$
$10\frac{3}{5} \div 3\frac{4}{7} =$
$8\frac{3}{7} \div 2\frac{3}{8} =$
$9\frac{5}{9} \div 1\frac{5}{7} =$
$6\frac{6}{7} \div 2\frac{1}{2} =$
$4\frac{7}{8} \div 3\frac{6}{7} =$
$11\frac{1}{9} \div 10\frac{1}{4} =$

$10\frac{1}{5} \div 3\frac{1}{4} =$
$9\frac{6}{7} \div 2\frac{5}{6} =$
$8\frac{5}{8} \div 6\frac{5}{6} =$
$4\frac{5}{9} \div 3\frac{4}{5} =$
$\frac{4}{3} \div \frac{4}{7} =$
$\frac{5}{8} \div \frac{5}{3} =$
$3\frac{3}{10} \div 2\frac{2}{7} =$

ANALYTICAL SOLUTIONS AND SHORT PROCESSES

Note 5.—This chapter completes the development of the language and processes of common fractions. The methods of solutions, excepting the unexplained and untalked-about method that the pupil discovered for finding in advance the results of operations in partition and multiplication as suggested in Chapter XXIII, Note, and Chapter XXIV, Note 4, have been in the long, the analytical forms. No suggestion of short processes has been made.

Short forms of solutions, excepting those in addition and subtraction of fractions outlined in Chapter XXVI, and the statements of the processes, the "rules," which are an essential part of all short process-work, should not be used below the seventh grade. This gives a full additional year for frequent reviews of the common fraction operations in practice and applied problems. These reviews must be made to parallel the decimal fraction and other work of Grade VI. From the fact that short processes have not been introduced, this review work will all be in analytical form.

The aim in this long use of analysis in common fraction operations is to preserve the fraction imagery and to develop more fully the power to analyze. Short processes are almost without exception mechanical operations—operations whose forms carry no suggestion of the reasoning involved in their use. Their effect, when not accompanied by regular and positive oral analysis requirements, is to cause the individual to lose or fail to acquire the imagery of the subject or operation and the power to verify solutions by orderly reasoning processes. It is for the sake of longer and wider experience in analysis and the self-reliance in computations that power to analyze tends to give, that short methods of almost every kind should be postponed to that grade. Their use even then should not be to the exclusion of analytical work in each subject.

Power to add, subtract, multiply, and divide with fractional numbers or with integers leaves the individual with little power practically unless there is with it the tendency and the power to submit operations to the test of logic.

Summary.

1. Object work.
   Exercises to develop the terms simple, fraction and complex fractions—
      (1) Applying the terms to concrete and

to written expressions (*A*, *2 a* and *b*.)

(2) Observing in a series of quotient constructions that part-groups in quotients made up of both integral and fractional units are complex. (*A*, *2 c* and *d* and Note A.)

(3) Quotients must not have complex fractions. (*A*, *3 a*.)

Constructions avoid such part-groups by changing dividend and divisor to fractional form.

Such changes must always be to lowest common denominator. (*A*, *3 b*.)

Exercises in construction in which the child is left to his own resources to construct in a way to avoid quotient fractions that are complex. (*A*, *3 c* and *4*.)

2. Work without objects. (*B*.)

In these the pupil must avoid complex fractions in quotient part-groups. To accomplish this dividend and divisor must be in fractional form.

Time required for this development and the necessary review work, fifteen days. Total time on common fraction development, thirty-two weeks—160 days. From four to eight weeks should be given to additional reviews—thirty-six to forty weeks in all.

CHAPTER XXVI

EMPHASIS POINTS IN COMMON FRACTIONS

Topics:—What the Pupil Should Know—In Each of the Fundamental Operations. In Partition Applications—Family Purchases Problems. Small Trade Problems

A. Aim

When the development work in any subject in arithmetic has been completed, the teacher should make a summary of its essential features as a guide in the reviews that must be given at frequent intervals later. The aim of this chapter is to outline for the teacher the kinds of review exercises that should follow the completion of common fraction development and that should be given from time to time each year during the remainder of the elementary school period. It is the answer to the questions, What should the child know in common fraction processes? What forms of solutions or computations must the pupil be able to make quickly, confidently, and accurately in order to meet the demands of practical life?

Fraction exercises under each of the fundamental operations may be arranged in three classes according to the forms in which the solutions are made:

Written—those in which the solutions are fully written out.

Partially written—those in which the pupil makes the solutions mentally, writing the results only.

Mental—those in which no part is written.

B. Addition

1. Written Solutions

a. MIXED NUMBER

(1) VERTICAL FORM.—Exercises in the addition of mixed numbers in the vertical form. This form of adding is common and practical. It should be the pupil's usual form of solution with mixed addends unless directed to use the horizontal.

$$\begin{array}{ccc} 3\frac{1}{2} & 5\frac{7}{8} & 15\frac{5}{9} \\ 4\frac{2}{3} & 3\frac{5}{7} & 14\frac{2}{7} \\ \text{—} & \text{—} & \underline{13\frac{2}{3}} \end{array}$$

(2) HORIZONTAL FORM.—Exercises in adding mixed numbers in horizontal form. Pupils will use this less than the vertical form in applied work, but the reviews should include a large amount of work in which this form is required. Pupils should use this form enough so that there will be no temptation to rewrite a horizontal dictation into vertical form unless the dictation so requires.

$2\frac{1}{2} + 3\frac{5}{6} + 4\frac{4}{5} = 2\frac{15}{30} + 3\frac{25}{30} + 4\frac{24}{30} = 11\frac{2}{15}$; or,

$3\frac{1}{3} + 2\frac{5}{8} + 1\frac{3}{4} = 7\frac{17}{24}$.

$3\frac{8}{24} + 2\frac{15}{24} + 1\frac{18}{24} = 7\frac{17}{24}$.

The very large proportion of the fraction additions in practical life are with fractions whose denominators are from 2 to 12.

This review work in the addition of mixed numbers should be very largely with such fractions. Fraction work with higher denominators may be postponed until after the highest common divisor and least common multiple have been developed.

b. FRACTIONS

Exercises in adding fractions when the operations involve common denominators so high as to make it inadvisable, in the interests of rapid and accurate work, to use partially written or mental solutions.

$\frac{5}{6} + \frac{3}{7} + \frac{2}{3} =$

$\frac{7}{8} + \frac{5}{9} + \frac{1}{6} =$

$\frac{5}{12} + \frac{5}{9} + \frac{5}{8} =$

$\frac{3}{5} + \frac{11}{12} =$

$\frac{6}{7} + \frac{7}{8} =$

$\frac{4}{5} + \frac{5}{7} =$

2. *Partially Written Solutions*

a. MIXED NUMBERS

(1) HORIZONTAL FORM.—The exercises of this class are those in which the addition of both integral and fractional parts may be made mentally, the child writing the result only.

The integers must, therefore, be small numbers and the fractional parts fractions of low denominators.

$2\frac{1}{2}+3\frac{1}{3}=$* $\qquad 3\frac{1}{2}+2\frac{7}{8}=$

$1\frac{3}{4}+1\frac{2}{3}= \qquad 4\frac{4}{5}+1\frac{1}{2}=$

$4\frac{1}{4}+1\frac{1}{6}= \qquad 2\frac{5}{6}+1\frac{1}{2}=$

(2) VERTICAL FORM.—What was suggested above for horizontal additions applies here. The exercises should be made up of low integral units and fractions having low denominators.

b. FRACTIONS

Exercises of this class are very practical and should be given an important place in review work. High common denominators and high numerators should be avoided. The aims are to make a large part of the work in each solution mental and to develop a growing degree of rapidity. Accurate results are first in importance. Accuracy must not be sacrificed in efforts toward *rapid* work.

$\frac{7}{8}+\frac{5}{6}=\frac{41}{24}=1\frac{17}{24} \qquad \frac{5}{9}+\frac{11}{12}=\frac{53}{36}=1\frac{17}{36}$

$\frac{2}{3}+\frac{3}{4}=1\frac{5}{12} \qquad \frac{3}{10}+\frac{2}{3}=\frac{29}{30}$

$\frac{5}{9}+\frac{5}{6}=1\frac{7}{18} \qquad \frac{3}{4}+\frac{5}{9}=\frac{47}{36}=1\frac{11}{36}$

---

* The teacher will understand that work of this kind is more difficult and should come after partially written work in which the addends are fractions.

3. Mental Solutions

This class of solutions differs from the "written" and the "partially written" forms in that the dictations are *oral* and the solutions mental with results stated orally. The work is very practical and much of it should be done. If mixed numbers are attempted with pupils who have become proficient in the work with fractions, the integral parts should be of low numbers. The special aim should be the accurate and rapid addition of fractions.

To be dictated orally:

$\frac{1}{3}+\frac{1}{2}=$
$\frac{2}{5}+\frac{9}{10}=$
$\frac{1}{4}+\frac{1}{3}=$
$\frac{5}{6}+\frac{1}{4}=$
$\frac{1}{6}+\frac{1}{2}=$
$\frac{5}{12}+\frac{3}{4}=$
$\frac{1}{2}+\frac{7}{8}=$
$\frac{3}{4}+\frac{3}{8}=$
$\frac{1}{3}+\frac{3}{8}=$

C. Subtraction

The suggestions for addition in *B* apply to subtraction. The vertical form of solutions with mixed numbers is specially important as training for computations in general, written, applied problems. That is the practical form with mixed numbers.

The "partially written" work after the general plan as that for addition should be given a very large place in the reviews because of its training for both written and mental work.

D. Multiplication

1. *Written Solutions*

a. MIXED NUMBERS

(1) VERTICAL FORM.—The practical and simple form of solutions with mixed numbers is the vertical. Very much work should be given in it in the reviews as well as during the development period. The large use of vertical solutions brings the pupil to recognize that it is simple, and that with it his work is more accurate and more rapid.

(2) HORIZONTAL FORM.—It will be noted that in this form of solution in multiplication the operation is by reducing the mixed number or mixed numbers to fractional form and then proceeding as with fractions. Nothing need be said here, therefore, except that the suggestions in *b* below apply here. The child should have much work in horizontal solutions with mixed numbers, but in *all* such exercises the dictation should state specifically that the solution is to be in horizontal form. It should be the general understanding with the pupils that, in the absence of directions as to the form, the solutions must be vertical.

b. FRACTIONS

It was suggested in *a* above that under the horizontal form of solutions in multiplication of mixed numbers the factors are reduced to fractional form. The reviews of horizontal work, therefore, are also reviews of the multiplication of *fractions*.

All Solutions Analytical.*

All work in multiplying one fraction by another (excepting that that is done later under "short methods") must be in the analytical form. Chapter XXIV (*A*, *2* and Note 3). This line of review work is of great importance on account of its training in general analysis. Very much of it should be given.

2. Partially Written Solutions

This is the form of work that is outlined in Chapter XXIV (*A*, *4*). It continues the valuable analytical work of the written solutions (*1 b*, above), but on account of the abridged form of the solutions is specially valuable as training in practical multiplication. Much attention must be given to this. The work is entirely in horizontal form.

3. Mental Solutions

These solutions should be in the *form* as outlined in Chapter XXIV (*A*, *4*). The exer-

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[*] The solutions in which the pupil finds the product by multiplying the numerators together and the denominators together belong to the "short methods" later.

cises should not involve the use of high denominators. The aim is full mental analysis, and fractions of low denominators are as valuable for analysis training as the higher forms. The lower fractions, however, offer less *computation* difficulties thus permitting the pupil to devote his thought more fully to the reasoning process. This mental work should be given much attention. (See footnote, p. 430.)

$\frac{3}{4}\times\frac{5}{7}=$ $\frac{5}{8}\times\frac{2}{3}=$ $\frac{2}{5}\times\frac{2}{9}=$ $\frac{7}{3}\times\frac{7}{4}=$

$\frac{2}{3}\times\frac{7}{8}=$ $\frac{5}{6}\times\frac{5}{9}=$ $\frac{5}{12}\times\frac{5}{2}=$ $\frac{6}{5}\times\frac{3}{5}=$

E. Partition

Partition is applied in so many ways in the large and in the smaller affairs of life as well as in such a variety of forms that the subject requires special attention in school work.

1. *Written Solutions*

a. THE PARTITION OF FRACTIONS

The reviews in fraction work must be wholly analytical. The shorter solution form—the product of the numerators and the product of the denominators—belongs to the later work in "short methods." The form of the written work should be that in Chapter XXIII (*B*, *2 b*).

$\frac{3}{5}$ of $\frac{7}{8}=$ $\frac{12}{5}$ of $\frac{2}{7}=$ $\frac{18}{4}$ of $\frac{2}{7}=$

$\frac{6}{7}$ of $\frac{6}{5}=$ $\frac{8}{3}$ of $\frac{14}{9}=$ $\frac{2}{10}$ of $\frac{15}{4}=$

b. THE PARTITION OF WHOLES

(1) All-Fraction Form of Solutions.

These solutions should not be separated from the "part-fraction" work. The dictations should be given together, the pupil to use the "all-fraction" solution only when the other is not possible. The choice of the method of solution in this respect should be a part of the pupil's problem.

(2) Part-Fraction Forms of Solutions.

This form can be used of course only with partitions in which the dividend is numerically larger than the denominator of the partitioning fraction. It must be used in all such problems unless the teacher specifically directs the solution in the all-fraction form.

$\frac{2}{3}$ of 2 = $\frac{7}{8}$ of 10 = $\frac{5}{12}$ of 15 = $\frac{4}{9}$ of 7 =
$\frac{5}{6}$ of 5 = $\frac{3}{4}$ of 14 = $\frac{7}{12}$ of 20 = $\frac{4}{7}$ of 3 =
$\frac{4}{9}$ of 12 = $\frac{5}{16}$ of 20 = $\frac{3}{5}$ of 7 = $\frac{5}{9}$ of 15 =

The pupil should have very much of this part-fraction work with low numbers. It is a form of work that touches closely his daily life. A piece of cheese, for illustration, weighing 22 oz. at 28¢ per pound is a problem for part-fraction computation—

28¢ + $\frac{3}{8}$ of 28¢ =

If the purchase is a piece of steak at 43¢ per lb. weighing 28 oz. the problem becomes—

43¢ + $\frac{3}{4}$ of 43¢ =

These are illustrations of very common problems and the pupil should be so well trained in all-fraction and part-fraction written work that he will discover the application of the latter form of solution and use it whenever problems with which it may be used come to him for mental solution.

c. THE PARTITION OF MIXED NUMBERS

These are exercises in which the dividends are mixed numbers. The dictations should compel the pupil to choose between the "all-fraction" and "part-fraction" forms of solutions. The forms of solutions are shown in Chapter XXIII (*E 1*, *c* (*1*) and (*2*).)

$\frac{2}{3}$ of $5\frac{1}{2} =$ $\frac{7}{8}$ of $9\frac{1}{4} =$ $\frac{5}{8}$ of $18\frac{4}{5} =$
$\frac{5}{6}$ of $3\frac{2}{3} =$ $\frac{12}{7}$ of $1\frac{2}{3} =$ $\frac{9}{10}$ of $5\frac{1}{3} =$

2. *Mental Solutions*

These are exercises like those in *1* above excepting that the work is wholly mental.

The *dictations* in this work should be *written*.

F. Special Partition Work

The following exercises connected more or less closely with partition are given for the sake of the training in analysis and in the "equation." They should have a large place in the review work because of the importance of such work later—in decimals and percentage.

1. *The Analysis*

a. Written Analysis

$\frac{5}{7}$ of 16 =
$\frac{1}{7}$ of 16 = $2\frac{2}{7}$
$\frac{5}{7}$ of 16 = $5 \times 2\frac{2}{7} = 11\frac{3}{7}$

$\frac{3}{8}$ of 22 =
$\frac{1}{8}$ of 22 = $2\frac{3}{4}$
$\frac{3}{8}$ of 22 = $3 \times 2\frac{3}{4} = 8\frac{1}{4}$.

Exercises:—

$\frac{3}{4}$ of 11 =
$\frac{2}{5}$ of 13 =
$\frac{6}{10}$ of 22 =
$\frac{5}{4}$ of 9 =
$\frac{6}{9}$ of 4 =
$\frac{5}{3}$ of 2 =
$\frac{5}{2}$ of 13 =
$\frac{7}{8}$ of 18 =
$\frac{4}{3}$ of 8 =

Questions on the Analysis.

After the solutions the teacher should begin to question the pupils on the number of steps in the solution, the operation in the first step, and the operation in the second step.

The questions are as follows:

On $\frac{5}{7}$ of 16 =

"How many steps in the operation?"

("Two, finding $\frac{1}{7}$ and then finding $\frac{5}{7}$.")

"What was the operation in the first step?"

("Dividing by 7, the denominator, to find $\frac{1}{7}$.")

"What was the operation in the second step?"

("Multiplying by 5, the numerator, to find $\frac{5}{7}$.")

These questions should thereafter be asked on each solution of this kind. The questions are asked *after* the solution.

Later the teacher should begin to question *before* the operations—

"How many steps will there be?"

"What will be the operation in the first step?"

"What will be the operation in the second step?"

This questioning will assist in making *mental* analyses of this kind later.

b. Mental Analyses

After three or four dozen solutions of this kind have been made with the questioning after each, the teacher should give written dictations for *mental* analyses. Much should be made of this mental work. Many problems should be given with questions upon each. Exercises of this kind should be given throughout the elementary school period.

2. *The Equation*

a. Written Equation Analysis

$\frac{3}{8}$ of — $= 12$
$\frac{1}{8}$ of — $= \frac{1}{3}$ of $12 = 4$
$\frac{8}{8}$ of — $= 8 \times 4 = 32$

$\frac{4}{3}$ of — $= 13$
$\frac{1}{3}$ of — $= \frac{1}{4}$ of $13 = 3\frac{1}{4}$
$\frac{3}{3}$ of — $= 3 \times 3\frac{1}{4} = 9\frac{3}{4}$

Exercises:—

$\frac{2}{3}$ of — = 11

$\frac{9}{8}$ of — = 17

$\frac{5}{4}$ of — = 19

$\frac{5}{9}$ of — = 21

$\frac{4}{3}$ of — = 22

$\frac{12}{5}$ of — = 25

$\frac{11}{7}$ of — = 16

$\frac{7}{11}$ of — = 16

$\frac{9}{2}$ of — = 17

Questions on the Equation Work.

Questions should be asked on each solution —those used in *1* above. The questions at first are after the solution. Later, they are asked beforehand.

b. Mental Equation Analyses

The suggestions as to mental analysis work in *1 b* above apply here. Many problems should be given for mental analysis and the work should continue throughout the remainder of the elementary school period.

3. Short Methods

a. Shortened All-Fraction Solutions

These exercises are to be solved by the all-fraction method. The solutions are written. The aim is to give a short written form that the pupil will be able to use later for short mental solutions. The teacher will *show* the pupil the solution form, at the same time writing it as a parallel to the usual analysis form. There is no attempt at a development. The two parallel solutions are intended to show the pupil that the process is merely the analysis form shortened.

THE ANALYSIS FORM	THE PARALLEL SHORT FORM
$\frac{2}{3}$ of $7 = \frac{2}{3}$ of $\frac{21}{3} = \frac{14}{3} = 4\frac{2}{3}$	$\frac{2}{3}$ of $7 = \frac{1}{3}$ of $14 = 4\frac{2}{3}$.
$\frac{3}{4}$ of $5 = \frac{3}{4}$ of $\frac{20}{4} = \frac{15}{4} = 3\frac{3}{4}$	$\frac{3}{4}$ of $5 = \frac{1}{4}$ of $15 = 3\frac{3}{4}$.
$\frac{5}{8}$ of $3 = \frac{5}{8}$ of $\frac{24}{8} = \frac{15}{8} = 1\frac{7}{8}$	$\frac{5}{8}$ of $3 = \frac{1}{8}$ of $15 = 1\frac{7}{8}$.

EXERCISES FOR PARALLEL SOLUTIONS AND "SHORT FORM" WORK

$\frac{2}{5}$ of $7 =$
$\frac{3}{7}$ of $2 =$
$\frac{5}{12}$ of $8 =$
$\frac{7}{10}$ of $2 =$
$\frac{8}{5}$ of $4 =$
$\frac{4}{11}$ of $13 =$
$\frac{2}{5}$ of $11 =$
$\frac{5}{12}$ of $7 =$
$\frac{3}{15}$ of $13 =$

Very many of these short form solutions should be given. They will be the basis later of mental solutions of exercises in which the dividends are integral. After a few parallel solutions the short form should be used alone. Pupils will call it "the short form."

b. ALIQUOT-PART SOLUTIONS

(1) WHERE APPLICABLE.—Partition enters so largely into the computations in the economic life of the family that its application to family purchases should be a part of the child's number training.

A woman asks her butcher for a pound of steak of a certain kind. The piece cut off proves to weigh 15 oz. At 43¢ per pound the problem of cost is $\frac{15}{16}$ of 43¢. The purchaser may select a piece of meat from the show-case.

It weighs 2 lb., 5 oz. At 38¢ per pound the problem of the price is

$$2 \times 38¢ + \tfrac{5}{16} \text{ of } 38¢.$$

The problem may relate to a head of cabbage that is sold by the pound. It may relate to a dozen oranges priced at 16 oranges for a quarter. It may relate to 5 articles which are priced at 3 for 25¢.

These illustrations show how widely partition applies in our everyday purchases. They also suggest a very valuable and practical line of "partially written" computations—computations that may, after sufficient experience in written work, be solved by many children mentally.

(2) THE METHODS.—To make solutions of this kind practical, there must be found a short form of work based upon reasoning rather than upon any possible formulas.

The simplest method, the only method under which solutions may be "reasoned out," is that by aliquot parts. This method is applied in so many ways that no attempt will be made here to do more than to suggest the general forms of such solutions. The work must at first be in the "partially written" form, the purely mental solutions being introduced gradually with problems that do not involve mixed number quotients or partial quotients.

(3) Forms of Solutions.—Exact Quotients

12 oz. at 43¢ per lb. . . . . $\frac{3}{4}$ of 40¢ + $\frac{3}{4}$ of 3¢
$\frac{3}{4}$ of 40¢. . . . . . . .30 ¢
$\frac{3}{4}$ of 3¢. . . . . $2\frac{1}{4}$¢
$32\frac{1}{4}$¢

9 oz. at 43¢ lb. . . . . $\frac{1}{2}$ of 43¢ = $21\frac{1}{2}$ ¢
$\frac{1}{16}$ of 43¢ = $2\frac{11}{16}$¢
$24\frac{3}{16}$¢

10 oz. at 45¢ lb. . . . .$\frac{1}{2}$ of 45¢ = $22\frac{1}{2}$¢
$\frac{1}{8}$ of 45¢ = $5\frac{5}{8}$¢
$28\frac{1}{8}$¢

11 oz. at 45¢ lb. . . . . $\frac{3}{4}$ of 45¢ = $33\frac{3}{4}$¢
$\frac{1}{16}$ of 45¢ = $2\frac{13}{16}$¢
$30\frac{15}{16}$¢

15 oz. at 41¢ lb. . . . . 1 lb. = 41 ¢
$\frac{1}{16}$ of 41¢ = $2\frac{9}{16}$¢
$38\frac{7}{16}$¢

14 oz. at 38¢ lb. . . . . .1 lb. . . . . . .38 ¢
$\frac{1}{8}$ of 38¢ = $4\frac{3}{4}$¢
$33\frac{1}{4}$¢

13 oz. at 38¢ lb. . . . . . $\frac{3}{4}$ of 36¢ = ...27 ¢
$\frac{3}{4}$ of 2 = $\frac{1}{4}$ of 6 = $1\frac{1}{2}$¢
$\frac{1}{16}$ of 38¢ = ... $2\frac{3}{8}$¢
$30\frac{7}{8}$¢

7 oz. at 38¢ lb. . . . . $\frac{1}{2}$ of 38¢ = 19¢

$\frac{1}{16}$ of 38¢ = $2\frac{3}{8}$¢

$16\frac{5}{8}$¢

Solutions in Outline (oz. in terms of lb.):

6 oz. is $\frac{1}{2}$ less $\frac{1}{8}$ (8 oz. − 2 oz.)
5 oz. is $\frac{1}{4}$ plus $\frac{1}{16}$ (4 oz. + 1 oz.)
5 oz. might be $\frac{3}{8} - \frac{1}{16}$ (6 oz. − 1 oz.)
3 oz. is $\frac{1}{4} - \frac{1}{16}$ (4 oz. − 1 oz.)
2 oz. is $\frac{1}{8}$

9 eggs at 23¢ doz. . . . . $\frac{1}{2}$ of 23¢ = $11\frac{1}{2}$¢

$\frac{1}{2}$ of $11\frac{1}{2}$¢ = $5\frac{3}{4}$¢

$17\frac{1}{4}$¢

9 at 23¢ doz. . . . $\frac{2}{3}$ of 23 = $\frac{2}{3}$ of 21 = ... 14 ¢

$\frac{2}{3}$ of 2 = $\frac{1}{3}$ of 4 = $1\frac{1}{3}$¢

$\frac{1}{12}$ of 23 = ... $1\frac{11}{12}$¢

$17\frac{1}{4}$¢

9 at 23¢. . . . . 1 doz. = 23 ¢

$\frac{1}{4}$ of 23 = $5\frac{3}{4}$¢

$17\frac{1}{4}$¢

11 at 27¢ doz. . . . . 1 doz. = 27 ¢

$\frac{1}{12}$ of 27 = $2\frac{1}{4}$¢

$24\frac{3}{4}$¢

10 at 27¢ doz. . . . . 1 doz. = 27 ¢

$\frac{1}{6}$ of 27¢ = $4\frac{1}{2}$¢

$22\frac{1}{2}$¢

Cost of 5 at 3 for 25¢ .... 6 cost .... 50 ¢
$\frac{1}{3}$ of 25¢ = $8\frac{1}{3}$¢
$41\frac{2}{3}$¢

At 16 for 25¢, cost of doz . . 16 .... 25 ¢
$\frac{1}{4}$ of 25¢ = $6\frac{1}{4}$¢
$18\frac{3}{4}$¢

1 lb. 5 oz. at 42¢ per lb. .... 1 lb. = 42 ¢
$\frac{1}{4}$ lb. = $10\frac{1}{2}$¢
$\frac{1}{16}$ lb. = $2\frac{5}{8}$¢
$55\frac{1}{8}$¢

The above solutions show the form of the work and the possibilities for mental solutions after considerable experience in rapid and independent written work. The three solutions of 9 eggs at 23¢ dozen show the possibilities in the way of individuality in the solutions. The aim in such work is independent thinking.

G. Partition Applied in Family Purchases

1. *Approximations in Dividends*

The outline given in *F* is the outline for *exact* dividends. In partition work applied to family purchases, however, the results should be approximations.

1 lb. 8 oz. of meat at 41¢ per lb. . . . . 1 lb. = . . . 41 ¢
$\frac{1}{2}$ of 41¢ = 20$\frac{1}{2}$¢
61$\frac{1}{2}$¢ = 62¢

14 oz. at 35¢ per lb. . . . 1 lb. = . . . 35 ¢
$\frac{1}{8}$ of 35¢ = 4$\frac{3}{8}$¢
30$\frac{5}{8}$¢ = 31¢

9 eggs at 35¢ doz. . $\frac{3}{4}$ of 32 = ... 24 ¢
$\frac{3}{4}$ of 3 = $\frac{1}{4}$ of 9 = 2$\frac{1}{4}$¢
26$\frac{1}{4}$¢ = 27¢*

At 3 for 35¢ cost of 5 . . . . 3 cost . . . . 35 ¢
$\frac{2}{3}$ of 35¢ = $\frac{1}{3}$ of 70¢ = 23$\frac{1}{3}$¢
58$\frac{1}{3}$ = 59¢

2. *How to Compute Approximations*

(i) 10 oz. at 33¢ per lb.
$\frac{1}{2}$ of 33¢ = 17¢ (8 oz.)
$\frac{1}{8}$ of 33¢ = 4¢ (2 oz.)
21¢

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* As this work relates to computations in connection with the small purchases for the kitchen, the practice in school exercises in the subject should be to give the seller in each solution the full benefit of the fraction. This means that if the exact cost of an article should be 42$\frac{1}{4}$ cents, the quotient should be called 43 cents. This is the common and perhaps regular practice in the retail trade where the business has to do with family food supplies.

(ii) 15 oz. at 29¢ per lb.

$$\begin{array}{lr} 1\text{ lb.} = \dots\dots\dots & 29\text{¢} \\ \frac{1}{16}\text{ lb.} = \frac{1}{16}\text{ of } 29\text{¢} = & \underline{1\text{¢}} \\ & 28\text{¢} \end{array}$$

(iii) 7 oz. at 31¢ per lb.

$$\begin{array}{lr} \frac{1}{2}\text{ of } 31\text{¢} = & 16\text{¢} \\ \frac{1}{16}\text{ of } 31\text{¢} = & \underline{2\text{¢}} \\ & 14\text{¢} \end{array}$$

(iv) 2 lb. 5 oz. at 37¢ per lb.

$$\begin{array}{lr} 2\text{ lb.} = \dots\dots\dots & 74\text{¢} \\ \frac{1}{4}\text{ lb.} = \frac{1}{4}\text{ of } 37\text{¢} = & 9\text{¢} \\ \frac{1}{16}\text{ lb.} = \frac{1}{16}\text{ of } 37\text{¢} = & \underline{3\text{¢}} \\ & 86\text{¢} \end{array}$$

3. *Practical Value of Approximations*

The greater part of the work in aliquot parts after the pupils acquire skill in the work with *exact* dividends should be in computing approximate results as in the retail stores. This is the form of work in partition that we regularly meet with in practical life. The aim, therefore, should be to develop rapidity in approximations—

(1) In "partially written" work.

(2) In mental solutions.

Because of its value in practical life, this form of partition work, first written and afterwards mental, should be given a very important place in number reviews throughout the remainder of the elementary school course.

The following are some of the classes of price units that should be recognized in preparing exercises in applied partition:

A certain number for 10¢;
A certain number for 5¢;
A certain number for 25¢;
A given price per dozen;
A given price per pound.

H. Partition in Small Trade—"Quarter's Worth," "Dime's Worth"

There is a large amount of small retail business in units of "quarter's worth," "dime's worth," "nickle's worth," etc. These problems involve partition approximations and should be made a part of the regular written and mental work in that subject.

At 30¢ per lb.—dime's worth.
  1 dime is $\frac{1}{3}$ of 30¢.
  $\frac{1}{3}$ of 16 oz. = $5\frac{1}{3}$ oz.—$5\frac{1}{4}$ oz.*
At 40¢ per lb.—5¢ worth.
  5¢ is $\frac{1}{8}$ of 40¢.
  $\frac{1}{8}$ of 16 oz. = 2 oz.

---

* The approximation in ounces will depend upon the scales. It is assumed that the ordinary counter scales will weigh to $\frac{1}{4}$ oz. In that case the nearest lower approximation to $\frac{1}{3}$ oz. would be $\frac{1}{4}$ oz.

At 40¢ per doz.—25¢ worth.
25¢ is $\frac{5}{8}$ of 40¢.
$\frac{5}{8}$ of 12 oz. $=\frac{1}{8}$ of $60 = 7\frac{1}{2} = 7$

At 25¢ per doz.—10¢ worth.
10¢ is $\frac{2}{5}$ of 25¢.
$\frac{2}{5}$ of $12 = \frac{1}{5}$ of $24 = 4\frac{4}{5} = 4$.

At 35¢ per lb.—25¢ worth.
25¢ is $\frac{5}{7}$ of 35¢.
$\frac{5}{7}$ of 16 oz. $=\frac{1}{7}$ of $80 = 11\frac{3}{7}$ oz. $= 11\frac{1}{4}$ oz.

At 6 for 25¢—10¢ worth.
10¢ $=\frac{2}{5}$ of 25¢.
$\frac{2}{5}$ of $6 = \frac{1}{5}$ of $12 = 2$.

The exercises in this form of applied partition should be at first written and afterward mental. This is valuable practical work and much of it should be given as long as the pupil attends school.

I. Division.

1. *Written Solutions*

After the development of division has been completed, the subject offers very few difficulties.

As this solution process is by reducing dividend and divisor to fractional form and to a common denominator and then dividing the numerator of the dividend by the numerator of the divisor, the review work consists simply of examples for practice. There should be much of this work but there is no need for classifying the exercises. The mixed number and the fraction division involve the same principle.

2. *Partially Written Solutions*

The pupil should have many exercises consisting of mixed numbers with low integral parts and fractions of low denominators; also exercises in which the dividend and divisor are fractions of low denominators.

In these solutions the pupil should write only the quotient, the reduction to common denominator and the division being mental. This is valuable work and there should be much of it.

3. *Mental Solutions*

There should be much work in wholly mental division—the *dictations written.* The exercises should consist of dictations like those in *2* but somewhat lower, perhaps, in the integers and the fractions used.

J. The Significance of the Fraction—A Partition or Division Operation

1. *Analysis of the Fraction*

(*a*) AIM.—There are three or four very common operations in fractions that for intelligent use require that the pupil shall be very familiar with the division and the partition significance of simple common fraction expressions. His attention should be specifically called, for illustration, to the fact that a fraction as $\frac{5}{7}$ may signify $5 \div 7$ or the partition process expressed in $\frac{1}{7}$ of 5. If we place before him the expression $5 \div 7 =$, he will immediately write the quotient $\frac{5}{7}$. If we give him the expression $\frac{1}{7}$ of 5, he will write the full analysis and quotient, $\frac{1}{7}$ of $5 = \frac{1}{7}$ of $\frac{35}{7} = \frac{5}{7}$. Nothing in his work, however, has caused him to associate closely $\frac{5}{7}$ with $5 \div 7$ or $\frac{5}{7}$ with $\frac{1}{7}$ of 5.

The aim of the work in this section is to teach him to associate fractions with their division and partition equivalents, and to use these equivalents, each at the proper time later, as the basis of proofs in the following operations:

(1) Reducing a complex common fraction to simple form.

$$\frac{3\frac{1}{3}}{4} = \frac{1}{4} \text{ of } 3\frac{1}{3} = \frac{1}{4} \text{ of } \frac{10}{3} = \frac{1}{4} \text{ of } \frac{20}{6} = \frac{5}{6}$$

$$\frac{3\frac{1}{3}}{4} = 3\frac{1}{3} \div 4 = \frac{10}{3} \div 4 = \frac{10}{3} \div \frac{12}{3} = \frac{10}{12} = \frac{5}{6}$$

(2) Reducing a common fraction to decimal form. (See Vol. III)

$$\frac{5}{8} = \frac{1}{8} \text{ of } 5 = \frac{1}{8} \text{ of } \frac{5000}{1000} = \frac{625}{1000} = .625$$

(3) Reducing a complex decimal fraction to a lower equivalent. (See Vol. III.)

$$.25\tfrac{1}{3} = .2\frac{5\frac{1}{3}}{10} = .2\tfrac{8}{15}$$

$$\frac{5\frac{1}{3}}{10} = \tfrac{1}{10} \text{ of } 5\tfrac{1}{3} = \tfrac{1}{10} \text{ of } \tfrac{16}{3} = \tfrac{1}{10} \text{ of } \tfrac{80}{15} = \tfrac{8}{15}$$

b. The Analysis

The teacher places before the pupil for analysis solution $\frac{1}{3}$ of 2.

The child's solution will be—

$\frac{1}{3}$ of $2 = \frac{1}{3}$ of $\frac{6}{3} = \frac{2}{3}$.

He will then be shown the following complete statement—

$\frac{1}{3}$ of $2 = \frac{1}{3}$ of $\frac{6}{3} = \frac{2}{3}$.

$\therefore$ $\frac{1}{3}$ of $2 = \frac{2}{3}$ and $\frac{2}{3} = \frac{1}{3}$ of 2.

Immediately following this, with the same fraction in order to associate the partition and the division equivalents, he is given the *division* expression—

$2 \div 3 =$

The pupil will write at once—

$2 \div 3 = \frac{2}{3}$

The corresponding complete statement is now taught—

$2 \div 3 = \frac{2}{3}$

$\therefore 2 \div 3 = \frac{2}{3}$ and $\frac{2}{3} = 2 \div 3$.

These analyses should be spoken of as "the partition equivalent" and "the division equivalent." The teacher must see that the impression must be left very strongly that *any* fraction may have a partition or a division equivalent.

Many exercises should be given in writing out the full analysis statements. This work prepares the pupil for the use of the analysis later when he comes to complex fractions and to decimals.

Exercises.

Partition Equivalents.

$\frac{2}{5}, \frac{5}{6}, \frac{8}{9}, \frac{12}{7}, \frac{15}{8}, \frac{4}{9}, \frac{10}{11}, \frac{6}{11}, \frac{5}{14}, \frac{3}{5}, \frac{5}{9}, \frac{8}{10}, \frac{7}{8}, \frac{8}{7}.$

Division equivalents.

$\frac{3}{5}, \frac{1}{6}, \frac{4}{9}, \frac{9}{4}, \frac{10}{12}, \frac{12}{10}, \frac{7}{8}, \frac{8}{7}, \frac{3}{2}, \frac{2}{3}, \frac{9}{5}, \frac{5}{9}, \frac{10}{8}, \frac{8}{7}.$

Exercises of this kind should be a part of the review work for many weeks. It will become *regular* work in complex fractions, and in decimals in Vol. III.

2. *Complex Fractions—Analysis Applied in Reductions*

In the development work in division of common fractions (See Chap. XXV) the pupil came in contact with the objective as well as the written complex fraction. In this section it will not be necessary to use the objects. He will consider the *written* complex fraction only.

The aim here is to require the pupil to apply to the complex fraction the analysis of the partition and the division equivalents with which he became familiar in *1 b*.

a. Forms of Statements

His forms of statements (there must be *two* statements, the partition and the division, for each of the fractions in the exercises) will be as follows:

$$\frac{3\frac{1}{2}}{5} = \frac{1}{5} \text{ of } 3\frac{1}{2} = \frac{1}{5} \text{ of } \frac{7}{2} = \frac{1}{5} \text{ of } \frac{35}{10} = \frac{7}{10}.$$

$$\frac{3\frac{1}{2}}{5} = 3\frac{1}{2} \div 5 = \frac{7}{2} \div \frac{10}{2} = \frac{7}{10}.$$

This work of reducing complex fractions to simple form is of great importance and he should have very much experience in the analysis.

After he has written out many fractions in the two-statement form as above, the teacher should give dictations in which the pupil shall analyze in but one form, the teacher directing him when she gives the dictation as to which form to use in the reduction.

b. Exercises in Complex Fraction Reductions

TWO-STATEMENT WORK

Fractions reduced by partition and afterward by division.

$$\frac{2\frac{1}{3}}{4},\ \frac{5\frac{2}{3}}{2},\ \frac{3\frac{3}{4}}{6},\ \frac{1\frac{1}{4}}{3},\ \frac{2\frac{3}{5}}{5},\ \frac{3\frac{3}{5}}{7},\ \frac{2\frac{3}{4}}{9},\ \frac{1\frac{1}{6}}{8}.$$

SINGLE STATEMENT WORK—PARTITION

Fractions reduced by partition.

$$\frac{4\frac{1}{4}}{2},\ \frac{3\frac{2}{3}}{5},\ \frac{5\frac{1}{3}}{6},\ \frac{4\frac{2}{3}}{8},\ \frac{1\frac{1}{2}}{8},\ \frac{2\frac{1}{7}}{5},\ \frac{1\frac{3}{7}}{9},\ \frac{4\frac{1}{2}}{3},\ \frac{5\frac{2}{5}}{12},\ \frac{3\frac{7}{8}}{4}.$$

SINGLE STATEMENT WORK—DIVISION

Fractions reduced by division.

$$\frac{1\frac{1}{2}}{5},\ \frac{2\frac{2}{3}}{6},\ \frac{3\frac{3}{4}}{5},\ \frac{3\frac{2}{3}}{3},\ \frac{8\frac{1}{3}}{10},\ \frac{7\frac{1}{3}}{11},\ \frac{9\frac{1}{3}}{9},\ \frac{5\frac{1}{2}}{9},\ \frac{1\frac{7}{8}}{3},\ \frac{2\frac{4}{5}}{2}.$$

SUMMARY.

Reviews of addition with equal emphasis upon written, partially written, and mental solutions. (*B.*)

Similar review work in subtraction. (*C.*)

Emphasis in multiplication (*D.*)—

- (*a*) Upon written vertical multiplication with mixed numbers.
- (*b*) Upon written horizontal multiplication with numbers in fractional form.

- (*c*) Upon multiplication analysis in all forms of solutions.
- (*d*) Upon "partially written" and "mental" analytical solutions using low denominators.

Emphasis in partition reviews (*E.*)—

- (*a*) Upon written partitions of wholes and mixed numbers in which part-fraction solutions are used when possible unless otherwise directed.
- (*b*) Upon mental partition with wholes and fractions.
- (*c*) Upon partition analysis and equations—written and mental.
- (*d*) Upon partition short methods (*F.*)—
  - (1) The shortened form.
  - (2) The aliquot parts form.
- (*e*) Upon partition applied (*G.*)—
  - (1) In family purchases.
  - (2) In small trade.

Emphasis in division reviews (*I.*)—

- (*a*) Upon written division of mixed numbers and fractions.
- (*b*) Upon partially written and mental division.

Analysis of the fraction (*J.*)—

- (*a*) Its partition equivalent.
- (*b*) Its division equivalent.
- (*c*) The complex fraction analyzed.

CHAPTER XXVII

SHORT PROCESSES AND RULES IN COMMON FRACTIONS

Topics:—The Short Processes in the Fundamental Operations. The Definitions of Fraction Terms. The Rules or Process Statements

A. Aims and General Principles

1. The aim of this chapter is to outline for the teacher the short processes in the five fundamental operations in common fractions and, in connection with such work, the definitions of the terms and the statements of the processes—these forming a very important part of common fraction instruction and training in oral and written English.

2. *Short Processes Deferred*

It is suggested that, because of the tendency after the introduction of short processes to neglect analysis (see Chapter XXV, Note 5), the work of this chapter, excepting that which relates to addition and subtraction, be deferred to the seventh school year and be made a part of the reviews of that grade. Under this plan fraction analysis will be continued as a part of the reviews that regularly parallel the work of decimal fraction training. This plan is very much to the child's advantage in that it gives him longer experience in analytical work thereby contributing to more fully established habits of orderly thinking.

3. Rapid Solutions

The general aim in short processes is timesaving. They may have an influence toward more rapid mental processes but they contribute no more in this respect than is contributed by analysis under equally energetic schoolroom leadership.

Rapidity in solutions is a relative term. Children of equal ability differ in their thought processes in this respect just as they differ in their rates of physical movements. Teachers must not expect equally rapid solutions from all, nor must they make time in solutions the sole basis of judgment of comparative power in number. There is a reasonable time within which a solution of a given kind is rapid, and judgment of power must be based upon accuracy within this time.

4. Rules or Process Statements

In the matter of verbal statements of processes—statements of what the processes are—which are generally spoken of under the name of "rules," no one thing aids more in the establishment of imagery than to develop, after the written work in any subject has been completed, statements, in oral and in written form, of what the numerical process is. Statements in good English answering questions like, How do we change a mixed number to fraction form? Or, How do we multiply one fraction by another? Or, How do we add fractions?, etc., are very valuable indeed in common fraction training and equally or even more valuable as subjects for oral and written composition. It trains to full and clear statements of what the computation acts are. It is practically objective work in oral and written language.

B. Addition

1. Two Short Processes

There are two short processes in written work in this subject, one in which the reductions to common denominator and the additions are entirely mental, the other in which the common denominator equivalents are written before the work of adding.

THE TWO PROCESSES ILLUSTRATED

These are illustrated in the following solutions:

1st. (*a*) $\frac{2}{3}+\frac{3}{4}=\frac{17}{12}=1\frac{5}{12}$; or, shorter still,
(*b*) $\frac{2}{3}+\frac{3}{4}=1\frac{5}{12}$.

2d.—$\frac{2}{3}+\frac{1}{7}+\frac{3}{4}=\frac{56}{84}+\frac{12}{84}+\frac{63}{84}=\frac{131}{84}=1\frac{47}{84}$.

In *1st*, the reduction to a common denominator and the addition are entirely mental, excepting (as in *a*) where the child is permitted to write the intermediate result ($\frac{17}{12}$) followed by the final result in simple form ($1\frac{5}{12}$).

In *2d*, the common denominator equivalents are written because it involves numbers so high that reductions and additions that are wholly mental (as in *1st*) are too difficult to be safely undertaken by anyone not trained by long practice in such work.

2. *The Shorter Solution—When Used*

The shorter process (*1st*) should be used in the solutions of exercises of low common denominators—denominators under 24, perhaps. Exercises should be given very frequently in sharp three-minute drills, most of the dictations to be given for mental solution entirely—no written work by the pupil. Other drills should be given in which the pupil merely writes the sum in its simplest form after the dictation, the computations being mental.

3. *Oral Work*

Oral work in class exercises is an important part of short process training. One pupil and then another goes to the blackboard to solve and complete a written dictation. This class exercise with its oral work should be as follows:

$\frac{1}{2}+\frac{4}{5}=$. (This represents the teacher's blackboard dictation.) The pupil with his finger on the $\frac{1}{2}$ says, "Five tenths and" (with finger on the $+$) "eight tenths" (with finger on the $\frac{4}{5}$) "are thirteen tenths. This is equal to one and three tenths" (writing $1\frac{3}{10}$ after the $\frac{1}{2}+\frac{4}{5}=$). The completed dictation is now $\frac{1}{2}+\frac{4}{5}=1\frac{3}{10}$.

4. *Illustrative Exercises*

SHORTER PROCESS

$\frac{1}{2}+\frac{2}{3}=$

$\frac{2}{3}+\frac{5}{6}=$

$\frac{3}{4}+\frac{5}{8}=$

$\frac{3}{4}+\frac{5}{6}=$

$\frac{3}{5}+\frac{1}{2}+\frac{3}{4}=$

$\frac{5}{6}+\frac{7}{9}=$

$\frac{3}{4}+\frac{5}{6}+\frac{1}{2}=$

$\frac{5}{8}+\frac{1}{2}+\frac{2}{3}=$

$\frac{1}{4}+\frac{1}{6}+\frac{3}{8}=$

$\frac{1}{2}+\frac{1}{4}+1\frac{1}{12}=$

LONGER PROCESS

$\frac{2}{5}+\frac{1}{2}+\frac{5}{6}=$

$\frac{5}{8}+\frac{4}{5}+\frac{3}{4}=$

$\frac{7}{12}+\frac{2}{7}+\frac{5}{6}=$

$\frac{5}{10}+\frac{5}{12}+\frac{3}{4}=$

$\frac{2}{3}+\frac{1}{2}+\frac{3}{15}=$

$\frac{7}{9}+\frac{1}{4}+\frac{5}{6}=$

$\frac{3}{7}+\frac{2}{3}+\frac{1}{2}=$

$\frac{7}{8}+\frac{5}{9}+\frac{3}{4}=$

$\frac{5}{16}+\frac{2}{3}+\frac{3}{8}=$

$\frac{1}{2}+\frac{5}{7}+\frac{1}{8}=$

5. *Definitions under Addition*

a. The terms to be defined under addition are as follows:

Fraction.
Proper fraction.
Improper fraction.
Mixed number.
Integer.
Numerator.
Denominator.
Terms of a fraction.
Complex fraction.
Sum.
Amount.
Addition.

b. The Forms of Questions on Definitions.

In asking for definitions, the teacher should use questions in all the forms in which they commonly occur, so far as possible. The questions on denominator, for illustration, should be—

Define denominator.

What is the denominator of a fraction?

What is meant by the denominator of a fraction?

The same question forms are used with reference to numerator.

The questions on *sum* should be—

What is meant by sum?

Define sum.

What is the result of an addition?

What do we call the result in addition?

The same general forms are to be used with amount.

c. The Definitions.

(1) A fraction is a number having one or a group of fractional units.

(2) A proper fraction is a fraction whose equivalents are less than a whole.

(3) An improper fraction is a fraction whose equivalents are equal to or greater than a whole.

(4) An improper fraction is a fraction whose equivalents, when expressed in lowest terms, are integers or mixed numbers.

(5) A mixed number is a number having one or more integral units and one or more fractional units of the same kind.

(6) An integer is a whole number.

(7) The numerator of a fraction is the number above the fraction line. It shows the number of units in the fraction.

(8) The denominator of a fraction is the number below the fraction line. It is the name of the fraction and shows into how many parts the whole has been divided.

(9) The terms of a fraction are the numerator and the denominator.

(10) A complex fraction is a fraction whose numerator or denominator is a fraction or a mixed number.

(11) A simple fraction is a fraction whose numerator and denominator are integral numbers.

(12) The sum is the result of the addition.

(13) Amount is the result of an addition.

(14) Addition is the process of finding the sum of two or more numbers.

Note.—The term common fraction cannot be introduced until after the term decimal fraction has been taught. A decimal fraction is a fraction whose denominator is some power of 10. It is not usually written. A common fraction is a fraction whose denominator is *written*. Manifestly, therefore, *common* fraction is a term that refers to a form in contrast with the *decimal* form. It must, therefore, be introduced after the decimal fraction form is understood.

6. *Rules in Connection with Addition*

The following rules or statements of operations are developed directly from the objective work. They express construction processes: How do we add fractions? To add fractions reduce them if necessary to a common denominator, add the numerators, and write the sum over the common denominator.

How do we reduce a mixed number to an improper fraction?

To reduce a mixed number to an improper fraction, multiply the integer by the denominator, add the numerator to the product, and write the sum over the denominator.

How do we reduce a mixed number to a fractional form?

To reduce a mixed number to a fractional form, multiply the integer by the denominator, etc.

How do we reduce an improper fraction to an integer or a mixed number?

To reduce an improper fraction to an integer or a mixed number, divide the numerator by the denominator.

How do we reduce a fraction to its lowest terms?

To reduce a fraction to its lowest terms, divide numerator and denominator by their highest common divisor.

NOTE.—This rule should not be used until after the term highest common divisor has been introduced. The pupil's process of reducing to lowest terms has been a matter of finding the terms of series—finding the lowest term. This and finding the common denominator have been mental, by inspection.

C. Subtraction

1. Two Short Processes

The short processes in this subject are in character similar to those in addition. The two forms are as follows:

1st. $\frac{5}{6}-\frac{3}{4}=\frac{1}{12}$.
2d. $\frac{7}{8}-\frac{5}{9}=\frac{63}{72}-\frac{40}{72}=\frac{23}{72}$.

The form in 1st should be used in all solutions having low common denominators; the 2d form, for solutions in which the numbers involved are too high for assured accuracy in results. These two forms are like those for addition in *B 1*.

Strong three-minute exercises from *written* dictations should be given frequently, first in written solutions in each of these two forms of arrangement. These should have in view the development of as high a degree of rapidity in solutions as is consistent with accuracy.

Exercises should be given afterwards in solutions from written dictations in which the reductions and subtractions are wholly mental.

2. *Oral Work*

The oral work is developed through class exercises as in addition, in which one and then another pupil goes to the blackboard to talk and write.

This oral work is conducted like corresponding work in addition, the pupil pointing to each term as the reduction and subtraction proceed.

This oral work is illustrated as follows:

$\frac{5}{8}-\frac{1}{6}=$. (This represents the written dictation.)

"Fifteen twenty-fourths" (pointing to the $\frac{5}{8}$), "minus four twenty-fourths" (pointing to the $\frac{1}{6}$), "leaves eleven twenty-fourths. The remainder is eleven twenty-fourths." The pupil then completes the dictation—

$\frac{5}{8}-\frac{1}{6}=\frac{11}{24}$.

3. *Illustrative Exercises*

SHORTER FORM

$\frac{5}{8}-\frac{1}{3}=$

$\frac{11}{12}-\frac{7}{8}=$

$\frac{3}{4}-\frac{7}{10}=$

$\frac{7}{8}-\frac{2}{3}=$

$\frac{2}{3} - \frac{3}{8} =$

$\frac{3}{8} - \frac{7}{20} =$

$\frac{7}{10} - \frac{1}{3} =$

$\frac{14}{15} - \frac{3}{10} =$

$\frac{11}{20} - \frac{3}{8} =$

$\frac{5}{6} - \frac{7}{10} =$

LONGER FORM

$\frac{7}{12} - \frac{2}{9} =$

$\frac{5}{6} - \frac{3}{7} =$

$\frac{11}{15} - \frac{3}{7} =$

$\frac{5}{9} - \frac{3}{8} =$

$\frac{11}{12} - \frac{5}{7} =$

$\frac{6}{7} - \frac{5}{9} =$

$\frac{11}{12} - \frac{10}{11} =$

$\frac{15}{16} - \frac{7}{10} =$

$\frac{7}{8} - \frac{2}{5} =$

$\frac{2}{3} - \frac{5}{11} =$

4. *Definitions under Subtraction*

a. The terms to be defined under subtraction are those given under *B* 5, and the following:

Minuend
Subtrahend
Remainder
Difference
Subtraction

b. Forms of Questions on Definitions.

These follow in general the forms for the corresponding questions under addition.

c. The Definitions.

(1) The minuend is the number from which we subtract.

(2) The subtrahend is the number to be subtracted.

(3) The remainder is the result after the subtraction.

(4) The difference is the result after the subtraction.

(5) Subtraction is the process of taking one number from another.

5. Rule in Connection with Subtraction

This rule (there is but one) should be added to the list in *B 6*, on account of subtraction.

How do we subtract one fraction from another?

To subtract one fraction from another, reduce them if necessary to a common denominator, subtract the numerator of the subtrahend from the numerator of the minuend, and write the remainder or difference over the common denominator.

D. Cancellation

1. Aim

This operation enters so largely into short process work that it is necessary to teach the pupil how to cancel and what the operation means. Canceling should not be a mechanical process. The pupil should know when he uses it what the nature of the operation is and what other operation or operations it represents. The work of canceling is very easily and quickly taken up by the pupil. The development work consists in leading him to recognize it as a division operation. The development is necessarily introduced through the use of cancellation in reducing fractions to lower and to lowest equivalents.

In the work of reducing fractions to lower and lowest terms in Chapter XX, the pupil talked about reducing by 8's, by 2's, by 6's, by 9's, by 12's, etc. His computation processes in such work were in fact dividing the terms of the fractions by 8, by 2, by 6, by 9, by 12, etc.

The pupil should now be required to put these processes into the language of "divide by." To do this, exercises in reduction (reviews) are given and this language is applied to them.

Change $\frac{9}{12}$ to lowest terms. ("$\frac{9}{12}=\frac{3}{4}$.")

"How did you change $\frac{9}{12}$ to $\frac{3}{4}$?"

("By reducing by 3's.")

"How did you reduce by 3's?"

("By dividing each term by 3.")

Change $\frac{12}{16}$ to the lowest term of the series. ("$\frac{12}{16}=\frac{3}{4}$.")

"How did you reduce it?"

("By reducing by 4's.")

"How did you reduce by 4's?"

("By dividing each term by 4.")

Through many exercises of this kind the language is brought into use. This being accomplished the pupil is ready for cancellation work.

2. *Single Cancellations*

a. MEANING

By single cancellation is meant a single division, dividing *once.*

The teacher should show the pupils the cancellation process in reducing fractions to a lower equivalent. In this she must use the language of "divide"—"I divide each term by 3," or, "I divide each term by 4," etc.

The teacher's work will be something as follows:

"I divided each term by 3 by canceling." The canceling process will thus be recognized as such division operation. The teacher's language in giving a fraction for reduction should be, Reduce this fraction to its lowest equivalent by canceling.

$$\frac{\overset{4}{\cancel{12}}}{\underset{5}{\cancel{15}}} = \frac{4}{5}$$

b. EXERCISES FOR REDUCTIONS

$\frac{9}{12} =$ $\frac{4}{6} =$ $\frac{12}{16} =$ $\frac{28}{35} =$ $\frac{9}{15} =$ $\frac{15}{20} =$

$\frac{6}{9} =$ $\frac{15}{20} =$ $\frac{14}{21} =$ $\frac{30}{40} =$ $\frac{14}{21} =$ $\frac{10}{12} =$

c. QUESTIONS

The following questions should be used with each reduction:

What did you do? ("I reduced the $\frac{9}{12}$")—or the $\frac{6}{9}$ or the $\frac{4}{6}$ as the case may be—("to its lowest equivalent by dividing each term by—.")

What was the common divisor of the terms?

What do we mean by canceling? ("By canceling, we mean that we reduce the fraction to a lower equivalent by dividing each term of the fraction by a common divisor.")

What did we divide by in this case?

d. ORAL WORK

Which equivalent have you? ("The lowest.")

After a few reductions followed by the questions, the pupil should be required to give the full oral work unassisted as he proceeds with the cancellation.

He goes to the board and talks as follows. ("I must reduce this fraction to its lowest equivalent by canceling. The terms of the fraction have a common divisor —. I divide by —. The equivalent is —. This is the lowest equivalent."

3. *Double Cancellations*

a. MEANING

By double cancellation we mean dividing each term of the fraction by a number and afterward by another number as when we reduce $\frac{12}{16}$ by canceling a 2 and then another 2.

b. AIM

The aim here is—

(1) To teach to cancel a second or a third time if necessary for reduction to lowest equivalent.

(2) To develop that *two* cancellations give the same equivalent as canceling once, using their product.

(3) To introduce the term highest common divisor.

c. Kind of Fractions

The fractions dictated for this development should be those having higher numerators and denominators. With such fractions there is more likelihood of the pupil's requiring a second cancellation to reduce to *lowest* equivalents.

d. Questions on Each Canceling Operation

The questions and the pupils' oral work are the same, practically, as suggested for single cancellations. (See *2 c* above.)

e. Form of Written Work

The following illustrates the form of the written work in cases where the pupil's first cancellation makes a second necessary in order to find the *lowest* equivalent:

$\frac{24}{60} =$

A pupil is sent to the blackboard to talk and write.

("I must reduce this fraction to its lowest equivalent by canceling. The common divisor of the fraction terms is 6. I divide each term by 6.")

$$\frac{\overset{4}{\cancel{24}}}{\underset{10}{\cancel{60}}} = \frac{4}{10}$$

(The dictation is now in this form.)

("The equivalent is $\frac{4}{10}$.")

"Have you its lowest equivalent?" ("No.")

The pupil is now told that he must reduce again. He does this, canceling a 2. The dictation on the blackboard is now in this form—

$$\frac{\overset{4}{\cancel{24}}}{\underset{10}{\cancel{60}}} = \frac{\overset{2}{\cancel{4}}}{\underset{5}{\cancel{10}}} = \frac{2}{5}$$

f. Oral Work

The oral work with the second cancellation should be as here shown. ("The common divisor of the two terms of the fraction is 2. I divide each term by 2. The equivalent is $\frac{2}{5}$. This is the lowest equivalent.")

g. Highest Common Divisor Introduced

The pupil should now be asked to write opposite this solution a second solution—a reduction in which only one cancellation is necessary. He is told that he must use the *highest* common divisor. The term will not require explanation. The pupil will get it by inference from his cancellation language.

The two solutions opposite each other will then appear on the blackboard as follows:

$$\frac{\overset{4}{\cancel{24}}}{\underset{10}{\cancel{60}}} = \frac{\overset{2}{\cancel{4}}}{\underset{5}{\cancel{10}}} = \frac{2}{5} \qquad \frac{\overset{2}{\cancel{24}}}{\underset{5}{\cancel{60}}} = \frac{2}{5}$$

He divides by 12. In making this division he, of course, must give the oral work. ("The highest common divisor of the two terms is 12. I divide each term by 12. The equivalent is $\frac{2}{5}$. This is the lowest equivalent.") This brings highest common divisor into use.

h. Special Questions on the Two Solutions

"In the first reduction, how many times and what did you cancel?"

("I canceled *twice*—a 6 and afterwards a 2.")

"In the second, how many *times* and *what* did you cancel?"

("I canceled *once*. I canceled a 12.")

"What was the highest common divisor of the terms?" ("12.")

"What single cancellation,* therefore, gives the same equivalent as canceling a 6 and then a 2?" ("Canceling 12.")

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* This question is of special importance because through its use with each double cancellation the pupil learns the fact that canceling a 6 and then a 3 is the same as canceling 18, that canceling 4 and then 7 is the same as canceling 28, etc. In other words, canceling two or more numbers is equivalent to canceling their product. It is a simple principle and the pupil should learn to recognize it.

i. Double Cancellations Developed

Several dictations for reductions should now be given. In each case where the pupil makes two cancellations, he should write opposite such solution another solution by one cancellation. The oral work in *f* and *g* and the questions in *h* must be used with each of these double solutions.

Oral Work.—After two or three of these fractions have been reduced to lowest equivalents in the two-solutions form and the questions upon each have been asked, one pupil and then another should be required to go to the blackboard with a fraction (as $\frac{36}{72}$), make the two solutions, and give the full oral work as follows without suggestion: ("In the first solution I canceled a 6 and afterward another 6. In the second, I canceled the highest common divisor* of the terms, 36. Each result is the lowest equivalent.") This full oral work must not do away with the questions suggested.

Exercises in double cancellations.

$\frac{42}{56} = \quad \frac{35}{105} = \quad \frac{18}{54} = \quad \frac{28}{98} = \quad \frac{52}{104} = \quad \frac{108}{132} =$

$\frac{36}{72} = \quad \frac{32}{160} = \quad \frac{24}{96} = \quad \frac{51}{102} = \quad \frac{44}{132} = \quad \frac{28}{84} =$

---

[*] The teacher must require that this form "I cancel the highest common divisor, etc.," be alternated with, "I divide each term by the highest common divisor."

j. QUESTIONS ON HIGHEST COMMON DIVISORS

What highest common divisor would give the same equivalent as a 6 and then a 3? As a 2 and then a 5? As a 5 and then a 2? As a 3 and then a 7? As a 7 and then a 3? As a 2 and then a 9? As a 9 and then a 2?

What two divisions would give the same equivalent as a highest common divisor 15? As a highest common divisor 9? As a highest common divisor 24? As a highest common divisor 22?

k. SHORTENING THE DOUBLE CANCELLATIONS

In the pupil's cancellations, thus far, he has made his second canceling by using the fraction result of the first:

$$\frac{\overset{4}{\cancel{12}}}{\underset{6}{\cancel{18}}} = \frac{\overset{2}{\cancel{4}}}{\underset{3}{\cancel{6}}} = \frac{2}{3}$$

He should now be told that the work may be shortened by making the second cancellation without rewriting:

$$\frac{\overset{\overset{1}{\cancel{4}}}{\cancel{28}}}{\underset{\underset{4}{\cancel{16}}}{\cancel{112}}} = \frac{1}{4} \qquad\qquad \frac{\overset{1}{\cancel{28}}}{\underset{4}{\cancel{112}}} = \frac{1}{4}$$

l. Oral Work

The pupil's oral work need not be changed with this form of solution because the intermediate fraction is apparent. When he cancels 7 in $\frac{28}{112}$, the result $\frac{4}{16}$ is before him in plain figures; just as, when afterwards he cancels 4, the result $\frac{1}{4}$ is before him. (See "Oral Work" on p. 471 and footnotes on pp. 470 and 471.)

m. Exercises in Shortened Cancellations

$\frac{32}{96} =$ $\frac{45}{105} =$ $\frac{54}{108} =$ $\frac{72}{120} =$

$\frac{39}{117} =$ $\frac{63}{252} =$ $\frac{27}{135} =$ $\frac{96}{112} =$

In these exercises the questions must be asked upon each. (See *2 c* and *3 h* above.) The oral work must not differ from that for the longer forms. (See *i* and footnote p. 471 above.)

4. Definitions

The terms are—

Cancellation.

Common divisor of the terms of a fraction.

Highest common divisor of the terms of a fraction.

The definitions—

Cancellation is the process of dividing the terms of a fraction by a common divisor.

A common divisor of the terms of a fraction is a number that will divide each of them evenly.

The highest common divisor of the terms of a fraction is the highest number that will divide each of them evenly.

E. Partition

1. *What the Short Process Is*

The child discovered in the regular work in partition (analysis) that he could anticipate the final result by taking the product of the numerators and the product of the denominators. (See Chapter XXIII, *C* Note.) This is the short process and requires little or no instruction excepting as to the *form* of the written work, the *oral* work, and the *cancellation*.

2. *The Oral Work of the Partition Process—The Rule*

The teacher writes a partition exercise for pupils to write the quotient direct—

$\frac{2}{5}$ of $\frac{3}{7}$ =

A pupil is asked to complete by writing the quotient—

$\frac{2}{5}$ of $\frac{3}{7} = \frac{6}{35}$.

With this and several others like it—partitions in which each quotient is in its lowest terms—the teacher develops the statement of the process—

("I multiplied the numerators together for the numerator of the quotient and the denominators together for the denominator of the quotient.")

The child must in every case make this statement of the process.

Exercises.

$\frac{2}{7}$ of $\frac{5}{9}$ = $\frac{7}{8}$ of $\frac{1}{2}$ = $\frac{6}{7}$ of $\frac{8}{11}$ =
$\frac{3}{8}$ of $\frac{5}{12}$ = $\frac{9}{10}$ of $\frac{3}{4}$ = $\frac{9}{2}$ of $\frac{3}{4}$ =
$\frac{2}{6}$ of $\frac{7}{9}$ = $\frac{9}{11}$ of $\frac{3}{7}$ = $\frac{11}{12}$ of $\frac{5}{9}$ =

3. *Canceling to Shorten the Process—Single Divisor*

a. AIM

The canceling work in partition is very quickly taken up, it is a matter of minutes. The work here is to develop that when we cancel in partition we merely reduce the quotient fraction to its lowest terms before we find it, that it is the same process of dividing numerator and denominator as we found it to be in *D*.

b. THE DEVELOPMENT

The teacher writes a partition expression on the blackboard—

$\frac{3}{4}$ of $\frac{2}{7}$ =

A pupil is asked to write what the partition quotient will be. He replies by completing the dictation—

$\frac{3}{4}$ of $\frac{2}{7} = \frac{6}{28} = \frac{3}{14}$

"How did you find the quotient, $\frac{6}{28}$?"

("I multiplied the numerators together for the numerator of the quotient, etc.")

"How did you reduce the quotient $\frac{6}{28}$ to its lowest equivalent $\frac{3}{14}$?"

("I divided each term by its highest common divisor, 2.")

"I will show you how to cancel the 2 beforehand—before we write the quotient."

$$\frac{3}{\underset{2}{\cancel{4}}} \text{ of } \frac{\overset{1}{\cancel{2}}}{7} = \frac{3}{14} \qquad \frac{3}{4} \text{ of } \frac{2}{7} = \frac{\overset{3}{\cancel{6}}}{\underset{14}{\cancel{28}}} = \frac{3}{14}$$

c. The Teacher's Statement

"I divided one of the numerators and one of the denominators by 2. The numerators are now 3 and 1; the denominators 2 and 7. I multiplied the numerators, 3 and 1, together for the numerator of the quotient and the denominators, 2 and 7, together for the denominator of the quotient."

A pupil is now sent to the blackboard to make the second solution, as above, and give the oral work in *D 2 d* as usual.

Several exercises should now be given for two-solution work as above, one pupil and then another going to the blackboard to write and talk. The oral work should be as in "The Teacher's Statement" above.

d. EXERCISES

$\frac{5}{8}$ of $\frac{12}{11}$ =
$\frac{7}{4}$ of $\frac{2}{5}$ =
$\frac{9}{12}$ of $\frac{5}{7}$ =
$\frac{15}{9}$ of $\frac{5}{11}$ =

$\frac{11}{3}$ of $\frac{4}{28}$ =
$\frac{21}{2}$ of $\frac{3}{14}$ =
$\frac{8}{12}$ of $\frac{5}{7}$ =
$\frac{2}{3}$of $\frac{21}{13}$ =

$\frac{3}{10}$ of $\frac{15}{4}$ =
$\frac{15}{20}$ of $\frac{7}{2}$ =
$\frac{8}{5}$ of $\frac{14}{21}$ =
$\frac{5}{9}$ of $\frac{15}{7}$ =

4. Questions

a. RELATING TO THE PARTITION PROCESS

$$\underset{2}{\frac{5}{\cancel{8}}} \text{ of } \frac{\overset{3}{\cancel{12}}}{11} = \frac{15}{22} \qquad \frac{5}{8} \text{ of } \frac{12}{11} = \frac{\overset{15}{\cancel{60}}}{\underset{22}{\cancel{88}}} = \frac{15}{22}$$

"What did you do when you divided the 8 and the 12 by 4?"

("I reduced the quotient fraction to its lowest equivalent, *beforehand.* I divided each term by 4.")

"What were the numerators after you cancelled?"

("5 and 3.")

"What were the denominators?"

("2 and 11.")

"How did you reduce the $\frac{60}{88}$ in the second solution to its lowest equivalent?"

("By dividing each term by its highest common divisor, 4.")

b. PARTITION TERMS—DIVIDEND, DIVISOR, QUOTIENT

The following additional questions upon each exercise should be asked in order to develop the partition terms:

"What does $\frac{5}{8}$ of $\frac{12}{11}$ tell us?" ("It tells us to divide $\frac{12}{11}$ into 8 equal groups, and show 5 of them.")

"What is the dividend?" ("The dividend is $\frac{12}{11}$.")

"What is the divisor?" ("The divisor is the denominator (8) of the partitioning fraction.")

"What is the result of the partition?" ("$\frac{15}{22}$.")

"What do we call this result?" ("The partition quotient.")

c. General Questions

These, now in the one form and again in the other, should be asked frequently in connection with all exercises in canceling in partition.

"What do we do when we cancel in partition?"

("We reduce the quotient fraction to its lowest equivalent, *beforehand*—before we write it.")

"What is cancellation in partition?"

("Cancellation in partition is the process of reducing the quotient fraction to its lowest equivalent, *beforehand*—before we write it.")

5. Double Cancellations—Two or More Divisors

The exercises in *3* above involved but one cancellation in each. In this step the exercises have two or more.

a. PLAN OF TEACHING

The teaching plan is to give the pupil an exercise for double solution as in *3* and, if he does not discover and use the second canceling opportunity, let him discover the need of such cancellation in order that his quotient fraction may be in its lowest form.

$\frac{12}{3}$ of $\frac{5}{8}$ =

If the pupil makes but one cancellation the two solutions may be as follows:

$$\frac{\overset{4}{\cancel{12}}}{\underset{1}{\cancel{3}}} \text{ of } \frac{5}{8} = \frac{20}{8} = 2\frac{4}{8} = 2\frac{1}{2} \qquad \frac{12}{3} \text{ of } \frac{5}{8} = \frac{\overset{5}{\cancel{60}}}{\underset{2}{\cancel{24}}} = \frac{5}{2} = 2\frac{1}{2}$$

This will show the quotient fraction $\frac{20}{8}$ not in its lowest terms. To most pupils this suggests the need of a second cancellation. All should rewrite the exercise and make the two cancellations—

$$\frac{\overset{\overset{1}{\cancel{4}}}{\cancel{12}}}{\underset{1}{\cancel{3}}} \text{ of } \frac{5}{\underset{2}{\cancel{8}}} = \frac{5}{2} = 2\frac{1}{2} \qquad \frac{12}{3} \text{ of } \frac{5}{8} = \frac{\overset{5}{\cancel{60}}}{\underset{2}{\cancel{24}}} = \frac{5}{2} = 2\frac{1}{2}$$

The questions in *4*, *a*, *b*, and *c* should be asked.

(The questioning should draw attention to the fact that the highest common divisor of the second solution is equal to the product of the common divisors used in the first solution.)

What are the numerators after the cancellations in the first solution?

("1 and 5.")

"What are the denominators?" ("1 and 2.")

"How do you find the quotient, $\frac{5}{2}$, in the first solution?"

("By multiplying the numerators, 1 and 5, together for the numerator of the quotient, and the denominators, 1 and 2, together for the denominator of the quotient.")

b. EXERCISES

$\frac{12}{18}$ of $\frac{6}{5}$ = $\frac{15}{6}$ of $\frac{6}{25}$ = $\frac{2}{9}$ of $\frac{12}{15}$ =
$\frac{7}{9}$ of $\frac{4}{21}$ = $\frac{2}{9}$ of $\frac{21}{8}$ = $\frac{9}{4}$ of $\frac{8}{14}$ =
$\frac{8}{10}$ of $\frac{6}{4}$ = $\frac{5}{7}$ of $\frac{28}{15}$ = $\frac{9}{11}$ of $\frac{33}{21}$ =

This completes the development of the partition short processes.

6. *Definitions*

The terms in partition are—

Partition
Quotient
Dividend
Divisor

The terms defined:

Partition is the process of dividing a quantity into equal groups.

Partition quotient—the result of a partition.

Partition dividend—the number that is to be divided into equal groups.

Partition divisor—the number showing how many groups the dividend is to be made into. It is the denominator of the partitioning fraction.

The teacher's questions in connection with definitions should show variety in form. To illustrate:

Define partition.

What is partition?

What do we mean by partition?

7. *Rules in Connection with Partition*

How do we solve a partition?

To solve a partition multiply the numerators together for the numerator of the quotient and the denominators together for the denominator of the quotient.

How do we reduce a fraction to a lower equivalent?

To reduce a fraction to a lower equivalent, divide the terms of the fraction by a common divisor.

How do we reduce a fraction to its lowest equivalent?

To reduce a fraction to its lowest equivalent divide the terms of the fraction by their highest common divisor.

8. Principles in Fractions

What effect does it have upon a fraction to divide the numerator and the denominator by a common divisor?

Dividing the numerator and the denominator of a fraction by a common divisor gives a lower equivalent.

What effect upon a fraction to divide both terms by their highest common divisor?

Dividing both terms of a fraction by their highest common divisor gives the lowest equivalent of the fraction.

F. Multiplication

1. Horizontal Solutions

(See Chapter XXIV).

The short processes in this subject are almost indentical with those of partition. This would naturally follow from the fact that the constructions in the two operations are similar in form (Chapters XXIII and XXIV).

No development, therefore, of multiplication will be necessary. The teacher will open the subject by giving exercises for solutions without preliminary explanations or suggestions, requiring solution forms and oral work as in partition and asking the same questions so far as they apply to multiplication.

The work should be similar:

In two-solution forms of work (*E*, *3 b*, *4 a*, and *5 a*);

In statements of the processes, excepting the use of the term *product* instead of *quotient* (*E*, *2* and *3 c*);

In the forms of canceling;

In stating what the new numerators and the new denominators are after canceling but before the product fraction is written (*E*, *3 c* and *5 a*);

In the application of the term common divisor to the number canceled in each case;

In the use of the expression, "I divide each term by its highest common divisor" (*E*, *3 b* and *4 a*) where the operation is by canceling, *afterwards;*

In noting the fact that the highest common divisor in each second solution is equal to the *product* of the divisors in the first solution (*E*, *5 a*);

In the oral work and the questions, so far as these are applicable to multiplication, excepting that *product* takes the place of quotient where the latter occurs (*E*, *2*, *3 b*, *4 a and c*, and *5 a*).

2. *Vertical Solution*

(See Chapter XXIV.)

The vertical solutions in multiplication are short processes—the shortest form for the multiplication of a mixed number by a mixed number.

3. Definitions

The terms are—

Multiplication.
Multiplicand.
Multiplier.
Product.

Multiplication is the process of taking a number a given number of times.

The multiplicand is the number to be taken.

The multiplier is the number that shows the number of times that the multiplicand is to be taken.

The product is the result of the multiplication.

4. The Rule for Multiplying Fractions

To multiply one fraction by another multiply the numerators together for the numerator of the product and the denominators together for the denominator of the product.

NOTE.—In all expressions in multiplication in *horizontal form* whether in abstract or applied work the second term in a multiplication expression must be considered the multiplicand. Consistency demands this.

In the expression 2 4's as used in Grade I, the very form of the expression indicates that it is the 4 that is being "taken" and that it is to be taken twice.

Following this 2×4 was read at first 2 4's. This interpretation made 4 the multiplicand. This made 4 concrete and the multiplicand.

This interpretation, if made then, must, for the sake of consistent language, be retained.

2 suits of clothes at \$25 each should be expressed 2×\$25=\$50.

25 hats at \$3 each should be expressed 25×\$3=\$75.

In all applied work the teacher should *insist* upon the concrete number in a multiplication expression being made the *second term* in the multiplication expression *in all cases.*

This does not mean that the other form of expression is wrong; but that, if the child starts out in the beginning of his number work with the second term *concrete*, it should remain so throughout his number training. We must be *consistent*. (See p. 389 Note 2.)

G. Division

1. Character of the Process

It is not necessary to outline the short process in this subject because, after the inversion of the divisor, the process is multiplication and the forms, oral work, and questions are those for that operation.

The pupil should be told what the short method is. It is not a process that admits of development. It is best treated as a mechanical process and given to the child as such.

2. Definitions

The terms are—

Division
Dividend
Divisor
Quotient

Division is the process of finding how many times one number is contained in another.

The dividend is the number to be divided.

The divisor is the number that we divide by.

The quotient is the result of the divison. It shows the number of times that the dividend will contain the divisor.

3. *Rule*

To divide one fraction by another, invert the divisor and multiply the numerators together for the numerator of the quotient and the denominators together for the denominator of the quotient.

APPENDIX

DENOMINATE NUMBERS

1. CHARACTER, AND PLACE IN THE CURRICULUM

This subject is one in which the pupil's work consists almost entirely in becoming familiar with the various units in our common system of weights and measures, in learning practically what each is used for, and in memorizing the long list of measuring units and facts. The operations involved, excepting those in surface and cubic measures, are very simple because they are applications of the simple operations which are developed with the child's first written applied number.

We except surface and cubic measures because in them there are other and greater problems than those of reductions ascending and descending. In these subjects the child must learn to compute quantity—quantity of surface and quantity of volume. These computations may be made objective by constructions, hence the possibilities of development methods of instruction in them. Under these conditions the two classes of denominate number units must be regarded as subjects to be given special attention—made special subjects in the number program of the elementary school.

The child should become acquainted in a practical way with a few carefully selected denominate number units in his first school year. The inch, the foot, the yard, inches in a foot, feet in a yard, the pint and the quart (if it is a community which patronizes milk peddlers), the peck (if the community is one in which vegetables are purchased in small quantities), days in the week, nickles in a dime, dimes in a dollar, etc. are among the measuring units and facts within the power of the primary pupil to understand practically. The units that are introduced in the first school year should be kept fresh and should be added to from year to year until the whole subject of English measuring units including surface and cubic measures has been completed. In this way denominate number teaching finds a place in every grade of the elementary school.

2. DENOMINATE NUMBERS CLASSIFIED AND DEFINED

A denominate number is a concrete number whose unit is quantitatively definite and is recognized by law or custom as a unit of measure.

Denominate numbers are of two classes—simple denominate numbers and compound denominate numbers.

A simple denominate number is one in which there is but one measuring unit.

5 feet has one measuring unit, foot.

3 hours has one measuring unit, hour.

The numbers are *simple*—simple denominate numbers.

A compound denominate number is one in which there are two or more measuring units.

4 gal., 2 qt. has two measuring units—gallon and quart. It is a denominate number. It is a *compound* denominate number.

3 yr., 7 mo., 2 da. has three measuring units—year, month, and day. It is a denominate number. It is a *compound* denominate number.

The teacher should require pupils to discriminate very carefully in the use of the terms denominate, simple denominate, and compound denominate numbers.

3. NOT A DEVELOPMENT SUBJECT

Denominate numbers, if we except surface and cubic measures, is not a development subject. In reductions, there is practically one simple operation in reductions descending and one in reductions ascending—operations which the pupil makes use of in his own very first written applied problem work. When, for illustration, he writes out the solution of the problem of the cost of 5 oranges at 4 cents each, his reasoning and arrangement are—

1 orange costs 4 cents.

5 oranges will cost 5×4 cents = 20 cents.

Reduction operations involve the very same form of reasoning, the same number of solution steps, and the same arrangement of written work. When, for illustration, the pupil reduces 5 gallons to pints his solution takes the form—

1 gallon has 8 pt.

5 gallons will have 5×8 pt. = 40 pt.

When the pupil writes out analytically the solution of the problem of the number of apples at 3 cents each that can be bought for 18 cents, his reasoning and arrangement are—

1 apple costs 3 cents.

To find the number for 18 cents—

18 cents ÷ 3 cents = 6 ∴ 6 apples.

(Mentally he asks, "18 cents has how many 3 cents?")

For comparison with this, when he reduces 64 oz. to lb. he uses the same line of reasoning and the same arrangement—

1 lb. = 16 oz.

To find the number of pounds

64 oz. ÷ 16 oz. = 4 ∴ 4 lb.

The two reduction problems used in this illustration are simple denominate numbers. The same principle applies, however, in reductions of compound denominate numbers. For illustration, let us reduce 3 yd., 2 ft. to inches. The form of solution that the pupil will use when such a problem is given him for the first time is—

$$\begin{array}{llr}
1 \text{ yd.} = 36 \text{ in.} & & \\
3 \text{ yd.} = 3 \times 36 \text{ in.} & = & 108 \text{ in.} \\
1 \text{ ft.} = 12 \text{ in.} & & \\
2 \text{ ft.} = 2 \times 12 \text{ in.} & = & 24 \text{ in.} \\
\hline
 & & 132 \text{ in.}
\end{array}$$

He makes two simple solutions and for his final result, the number of inches in the two quantities, he finds the sum of the results. The exact arrangement of the 24 in. under the 108 in. for convenience in adding must be suggested to him, otherwise the above solution will be his own.

This shows that there is nothing in written denominate number reductions that requires development because they involve only the simple processes and methods of arrangement that the pupil has already learned in construction work in written applied number. It shows also that denominate number reductions are naturally a part of the work in simple analysis and should form a part of the applied problem work of the grade where such work is first used. If made a part of the work of the third school year the teacher would see to it that the work given does not involve difficult operations.

4. REDUCTIONS—WHEN INTRODUCED AND THE FORMS OF STATEMENTS.

a. Oral Reductions

Mental work in reductions should be introduced when the pupil first begins to have measuring units. When, for illustration, he has been taught that 3 ft. = 1 yard, he should be given mental solutions without analysis—

How many ft. in 3 yd.? In 5 yd.?

How many yards in 9 ft.? In 12 ft.? In 6 ft.?

How many inches in 1 ft.? In 2 ft.?

With the introduction of 2 pts. = 1 qt. there are the reductions of quarts to pints and of pints to quarts. These mental reductions should involve at any given time all of the measuring units that the pupil has been taught unless the numbers are too difficult. The aim in this mental reduction work (which should be given regularly and should cover at any given period all of the measuring units that the child has learned at the time) should be (1) to assist in fixing the units in memory and (2) to give reasonable training in mental solutions. This mental work should not be in analytical form. Analysis is orderly arrangement of all solution steps. It is a development. This development must be made through work that appeals in part to the senses. In written analysis the child *sees* each step in written form. A second and a third analysis step, the one and then the other, may be worked out by the pupil independently of the teacher when the previous step or steps are thus before him. He is not compelled to carry them in memory, hence his effort, his thought, may be concentrated upon what the next step should be. The power of orderly oral analysis is the product of training in written analysis. Oral analysis as a preparation for written analysis is bad pedagogy. In this work of oral reductions the teacher does not inquire into the child's mental processes. She is interested entirely in the *product* of his processes. She may safely rely upon the products to indicate that the thinking is good or that it is bad.

The teacher's rule of action should be to carry the oral work with each denominate number reduction to the limit, and only to the limit, of the pupil's power to solve correctly with reasonable rapidity. This power will grow with well directed experience.

b. Written Reductions

Written reductions in denominate numbers should *always* be analytical in form. They should have for their aims experience (1) in fixing the units in memory and (2) in arrangement of work logically and the development of habits of thinking in an orderly way.

The development of written applied work should begin in the 2d or 3d school year—not later than the 3d.

After this development has reached a stage where objective work is no longer necessary—when the child is able to arrange an analysis in orderly form with or without the diagrams or other objective forms—the reductions of simple denominate numbers, involving only the units that have been learned at the time, may be introduced as forms of written applied problem work.

5. DESCENDING REDUCTIONS—THE NATURAL FORM

a. The Step by Step Form

The teacher's attention should be called here to the method of reducing compound denominate numbers by what we may call the step by step method.

Reduce 2 rd., 4 yd., 2 ft. to in.

The following illustrates it:

1 rd. = 5½ yd.
2 rd. = 2 × 5½ yd. = 11 yd.
11 yd. + 4 yd. = 15 yd.
1 yd. = 3 ft.
15 yd. = 15 × 3 ft. = 45 ft.
45 ft. + 2 ft. = 47 ft.
1 ft. = 12 in.
47 ft. = 47 × 12 in. = 564 in.

This is workmanlike but entirely artificial. It is a form of work that a child rarely, if ever, uses unless it has been given to him by a teacher or other person. It is not a child's way of procedure. The child should be taught the method, however, before he leaves school and should be required to make reductions under it, but not until after he has used his own natural method in many exercises.

b. The Natural Form

What the child's natural method is may be found by giving to a class (after the pupils have had large experience in reductions of *simple* denominate numbers) an easy compound denominate number for reduction—their very *first* problem in such numbers. The problem should be given without warning or suggestion. For illustration, the problem—

Reduce 6 yd., 2 ft. to inches.

The solution will be somewhat as follows:

$$\begin{array}{lr}
1 \text{ yd.} = 36 \text{ in.} & \\
6 \text{ yd.} = 6 \times 36 \text{ in.} = & 216 \text{ in.} \\
1 \text{ ft.} = 12 \text{ in.} & \\
2 \text{ ft.} = 2 \times 12 \text{ in.} = & 24 \text{ in.} \\ \cline{2-2}
 & 240 \text{ in.}
\end{array}$$

The pupils will find the 216 in. and 24 in. and will add them in their own way. The arrangement of these two results under each other far enough to the right so that no other part of the solution will obstruct is the only instruction that the teacher will find necessary to give.

This is the child's natural reduction method. All solutions will be made in this way unless some one instructs him in the step by step work.

This form of work is spontaneous, it is the child's way of thinking, it is clear, it is as brief as any, and it is easily read and understood.

6. Reduction Forms—Two Simple Operations

Aside from those in surface and cubic measures, the computations in denominate numbers involve but two forms of operations—reductions to lower and reductions to higher units.

a. Reductions Ascending

A reduction ascending problem is necessarily a *simple* denominate number. After the reduction we may have a compound denominate number result. The following illustrates the form of solution that the pupil naturally uses—the form that his work will take if he is not *instructed* in the subject:

Reduce 36 pt. to gal.

1 gal. = 8 pt.
36 pt. ÷ 8 pt. = 4½ ∴ 4½ gal.
4½ gal. = 4 gal., 2 qt.

He will change the pints to the denomination required. If he is simply forbidden to have a fraction in his result unless it can not be avoided, he will reduce any fraction that there may be in the quotient to lower integral units.

This method is his own. It is workmanlike. It is as brief as any other form of work. It should be accepted and encouraged.

The step by step form of ascending reduction may or may not be taught as the teacher may decide, but it should not be given until the pupil has had very much experience in reductions by his own method.

b. Reductions Descending

(1). With *Simple* Numbers.

The following solutions illustrate the statements in the analysis in a *simple* denominate number reduction:

(i). Reduce 4 yd. to inches.
1 yd. = 36 in.
4 yd. = 4 × 36 in. = 144 in.

(ii). Reduce 6 gal. to pt.
1 gal. = 8 pt.
6 gal. = 6 × 8 pt. = 48 pt.

Each has a single analysis, this analysis consisting of two related statements.

(2). With *Compound* Numbers.

The following solutions are representative of the two or three or more single analyses, each exactly like that in (i) or (ii), that make up a compound denominate number reduction in "natural form" solutions:

(iii) Reduce 5 rd., 4 yd. to ft.
1 rd. = 16½ ft.
5 rd. = 5 × 16½ ft. = 82½ ft.
1 yd. = 3 ft.
4 yd. = 4 × 3 ft. = 12 ft.
94½ ft. = 94 ft., 6 in.

(iv) Reduce 5 mi., 7 rd., 4 yd. to ft.

1 mi. = 5280 ft.
5 mi. = 5 × 5280 ft. = 26400 ft.
1 rd. = 16½ ft.
7 rd. = 7 × 16½ ft. = 115½ ft.
1 yd. = 3 ft.
4 yd. = 4 × 3 ft. = 12 ft.

26527½ ft. = 26527 ft., 6 in.

(3). How They Differ.

The difference between these operations is that that with a *simple* number has a single analysis, while that with a *compound* number has as many of these single analyses in the solution as there are different measuring units in the number to be reduced. In (i) the number is *simple* (one measuring unit, yard) and there is *one* analysis in the solution. The same is true of (ii). In (iii), there are *two* measuring units, rods and yards. In the solution there are *two* analyses,—one giving 82½ ft., the other 12 ft. The sum of these two single analyses is the result sought. In (iv), there are in the same way three measuring units and as a result *three* analyses in the solution. The sum of the results of these three analyses is the result desired.

The operations differ only in respect to the number of these simple analyses. The reduction of the *simple* number has *one*, that of the *compound* number has two or more.

The reductions, therefore, show one simple analysis form for all descending reductions and one simple analysis form for ascending reductions. In either case, if the solutions do not involve numbers that are too difficult, reductions may be a part of the applied problem work in any grade where applied work is given.

7. SUGGESTIONS ON SPECIAL MEASURES

a. Long Measure

It should be perfectly obvious that until the fraction *half* has been taken up in the fraction development work the units in linear measure must be limited to inches in a foot, feet in a yard, feet in a fathom (6), and feet in a pace.

The time, therefore, from the introduction of the first linear units (the first or second school year) to the completion of the table with yards and feet in a rod and rods in a mile would be three or four years. The fathom and the pace should have attention in the early grades. They are important because they are simple, every-day, measuring terms and should be considered items of necessary common information. Pupils should have practical experience in pacing off distances. They should have practice in pacing until they are able to take steps that are approximate yards.

After square inch, square foot, and square yard have been developed and brought into use in surface measure work there should be repeated tests of the pupil's linear measure and square measure imagery. These tests, in order that they may prove something as to whether the child discriminates clearly between a linear unit and the surface unit of the same name, must be made with questions which afford no suggestion of the answer. "Show me five different ways in which I can use linear measure on the board which forms the top of your desk." The child's answers should be explicit showing the definite points in each case from which and to which I may measure.

There is with all of the various measuring units the question, "What is this unit (or table) for?" The answer is, of course, something after this form—

"This unit (or table) is used for weighing common articles"; or, "This unit is used to measure liquids"; or, "This unit is used to measure the distance from one point to another"; etc. With linear measure, besides the usual form of statement—"To measure the distance, etc.," there should be the form, "To find *how far* it is from, etc." The most expressive language in which the pupil can answer a specific question as to where he would use linear measure would be in the form, "To find *how far* it is from here" (a definite point) "to here" (another definite point). This definite pointing out of the point *from which* and the point *to which* the pupil thinks he would measure is essential to proof that the child is thinking *length.*

Careless use of the signs ′ and ″ for feet and inches must not be permitted. They are proper symbols in linear measure but not for surface units of those names. 20′ means 20 linear feet. It is not an expression for surface or cubic feet. The symbols cannot be used with those units.

b. Time Measure

(1). A Memory Help.

Some of the facts under time measure are among the earliest that are introduced into the school work. The pupil in the first grade learns the number of days in the week and their names. The number of hours in the day, the number of months in the year, and the names of the months are facts that belong to a very early school period.

The number of days in the several months are not easy to memorize without some form of memory help. The old rhyme—

"Thirty days hath September,
April, June, and November.
All the rest have thirty-one
Excepting February, which alone
Hath twenty-eight; but one day more
We add to it one year in four,"

has served a very useful purpose with all of us and the teacher should not fail to give it to the pupils in this or in some other form at the proper time—when the number of days in the months is being memorized. The rhyme has other forms. There is no particular reason why one is better than another. The main point is that the (or a) rhyme is useful and the child should have it.

Questions on the number of days in the months should be in frequent use to compel the pupil to resort to the rhyme for the information that it contains.

(2). Computations in Time.

Exercises should be given as early as the 4th or 5th school year in computing time between dates. There are two methods of computing time. These may be spoken of as—

The time in exact days.

The time in calendar months.

By the latter is meant the method under which all months are supposed to be of the same length. The time, for illustration, from June 10 to July 10 is 1 month, the time from Jan. 25th to Feb. 25th is one month, and the time from Feb. 20th to March 20th is one month. In effect it assumes that there are 12 months in the year, that one month is just as long as any other month and that each month has 30 days.

(*a*). Computations by Calendar Months.

These are simply exercises in subtraction of compound denominate numbers. For illustration—

Find the time from April 20, 1916, to August 15, 1918.

1918 yr., 8 mo., 15 da.
1916 yr., 4 mo., 20 da.
———————————
2 yr., 3 mo., 25 da.

This is frequently spoken of as the method of "subtracting dates."

(*b*). Computations in Exact Days.

This must be separated into three or four operations if we would train the pupil to meet the problems in whatever form they come to him—

1st. Computing the time from one date to a later date.

Find the time from Jan. 5th to Oct. 25th, 1920.

2d. Computing a date that is a given number of days later than a given date.

What date is 70 days later than June 4th?

3d. Computing the length of time between a given date and an earlier date.

How long a time between July 4th and the previous March 29th?

4th. Computing a date that will fall a given number of days earlier than another given date.

What date is 80 da. earlier than Oct. 25th?

The computations under the first case should be carefully taught as follows:

Find the time from Jan. 20, 1916, to May 20, 1916.

Days left in Jan............	11 days
Days in Feb...............	29 days
Days in March............	31 days
Days in April.............	30 days
Days in May.............	20 days
	121 days.

The teacher will note that the first step in the computation is the *number of days left* in the first month of the period. This simplifies the computation for the other months.

The computation under the second case is made in a similar way—

What date falls 69 days later than August 17th?

Days left in August—14. . .55 days left.
Days in Sept. . . . . .30. . .25 days left.
Days in Oct. . . . . .25 ∴ Oct. 25th.

The computations in the third case are practically the same as those in the first case. The pupil would actually compute from the earlier to the later date.

The computations in the fourth case should be made on the same plan as those in the first case—

Find a date 75 d. earlier than April 3, 1920.
Days through April. . . . . . 3. . .72 days left.
Days through March. . . .31. . .41 days left.
Days through Feb. . . . . . . .29. . .12 days left.
Days back into Jan., 12. . 31 d.−12 d. = 19 d.
∴ Jan. 19th.

The proof of a computation of this kind is to count forward. In this case the pupil would count 75 d. forward from Jan. 19th.

Find a date 80 d. earlier than July 19th.
Days through July. . .19. . . .61 days left.
Days through June. .30. . . .31 days left.
Days through May. .31. . . . 0 days left.
∴ April 30th.

This work of computing time is proper for any grade in which interest is taught. The "calendar month" method and the first case in "exact days" are very simple and are proper for the grade preceding that in which interest for years is usually introduced.

c. Surface Measure

This and cubic measure are development subjects. They offer development possibilities in two respects—

1. In the relations of the units in each. There are, for illustration, reasons for 144 sq. in. in 1 sq. ft., 30½ sq. yd. in 1 sq. rd., 27 cu. ft. in 1 cu. yd., etc. These facts may be discovered by constructions.

2. In the computations.

These two tables are, among our English weights and measures, in a class by themselves in respect to computation, in that, besides instruction in the ordinary reductions ascending and descending, there are other special instruction problems—problems in computing quantity of surface and cubical quantity.

Surface measure is one of the difficult subjects in number teaching—not difficult in the computation work if the three distinct development stages are recognized and each is given its proper place in the course of study, but difficult in the matter of developing clear imagery. To train the child to discriminate positively between the surface of the object and the object itself is a task that puts any teacher's skill to severe test. There is no intelligent work in surface measure until the individual thinks surface when he talks about surface measure.

There are in surface measure development three stages of work, the second of which requires in the pupil a higher degree of maturity than the first, the third requiring a maturity of a still higher degree than the second—

1. The measurement of plane surfaces that have straight sides and square corners, the number of corners not being confined to four. This work belongs to the 4th and 5th school years.

2. The measurement of right triangles, rhomboidal forms, and oblique triangles in order. This work should be taken up one or two years later.

3. The measurement of the circle and the surfaces of cylinders, cones, pyramids, etc. This is grammar school work.

The units in the surface measure or the cubic measure table should not be given to the child until they are given in connection with the formal development of the subject.

d. Troy Weight

This is a grammar school subject on account of the fact that its principal interest to the ordinary individual is historical. It belongs to that part of the school course in which is taken up the history of our country from 1876 to 1896. No person can study intelligently the political history of this period without a knowledge of our gold and silver coinage system. That was a period of political struggles over the question of the basis of our currency, whether our money should be based upon a gold dollar of 25.8 grains, 9/10 pure gold, or upon a double standard—gold dollars of this weight and silver dollars weighing 412 ½ grains, the metal in the latter being 9/10 pure silver. The ratio of the weight of the silver to the gold dollar was expressed approximately in political phrase as "16 to 1." This expression would be empty, seriously indefinite, without a knowledge of the weights of the coins.

Again, not in connection with history, the terms "standard gold" and "standard silver" are in comparatively common commercial use. "Standard gold" refers to the United States standard for gold coins. It means that the article is made of gold of the United States gold coinage standard, 9/10 pure with 1/10 alloy (copper and silver). "Standard silver" has reference in the same way to the United States silver coinage standard, 9/10 pure silver with 1/10 alloy (copper). It means that the article is made of silver thus composed. The child should know these facts. They are matters of general information with which he should be familiar. The pupil should know that the alloy in our gold and silver coins is to give them hardness, that the pure metals are too soft to be made into money because the coins would wear down very rapidly in handling.

In connection with this instruction on the coins, the pupil should learn the Troy table. He should have experience with grain weights until he becomes able to *estimate* in such weights. He should be taught that 7000 grains is the equivalent of an avoirdupois pound. He should learn, by computing the number of grains in a Troy ounce (480) and the number in an avoirdupois ounce (1/16 of 7000), that the two "ounces" represent totally different weights. In the same way he should compare the Troy and the avoirdupois pounds—5760 gr. and 7000 gr. That a pound of iron or potatoes or butter and a pound of gold or silver are not equal weights is information that young people should have.*

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* Coinage Facts—That the Teacher May or May not Have Occasion to Use.

(1). On the Gold Dollar.

The gold dollar is not coined.

The dollar, if coined, would weigh 25.8 gr.

The 5-dollar gold piece weighs 5×25.8 gr., the other gold coins in proportion.

The pure gold in a dollar would weigh 9/10 of 25.8 gr. = 23.22 gr.

The gold coins have 9 parts (by weight) pure gold to 1 part alloy.

The alloy is copper and silver—19/20 pure copper, 1/20 silver.

(2). On the Silver Dollar.

The silver dollar weighs 412.5 gr.

The silver coins have 9 parts pure silver to 1 part pure copper.

The pure silver in a silver dollar is 9/10 of 412½ gr. = 371.25 gr.

(3). On Silver Halves, Quarters, and Dimes.

These are made of standard silver.

Each is slightly under-weight, the 50-cent piece, for illustration, weighing a little less than ½ of 412½ gr. This makes the exportation of the half-dollars, quarters, and dimes unprofitable.

(4). On the "Nickle."

This coin is made of ¼ nickle and ¾ copper.

(5). On the Cent.

This is bronze—19/20 copper, 1/20 tin and zinc.

e. Apothecaries Weight

(1). Comparative Values.

The work upon this table should follow that upon Troy weight. The apothecary units that are in common use are the grain, the ounce, and the pound. The pupil learns that the grain is the same unit (represents the same weight) as the grains in avoirdupois weight (7000 gr. = 1 lb.) and in Troy weight. By reducing a Troy ounce and an apothecary ounce to grains he finds that each has 480 grains, showing that the two ounces represent equal weights.

In a similar way the fact is demonstrated that the Troy pound and the apothecary pound represent equal weights, 5760 grains. The avoirdupois pound has 7000 grains—a much heavier weight than the Troy or apothecary pound. By finding the weight of the avoirdupois ounce (1/16 of 7000), proof is made that this ounce represents a weight that is less than the other ounces.

The aim in work on this table is almost purely informational. The pupil should learn that the scales upon which the units in this table are used are drug-store scales for weighing medicines.

(2). The Apothecary Symbols.

Sometimes teachers who find it desirable to teach the apothecary symbols have difficulty in doing so. These signs, however, are simple if the teacher will analyze them.

The scruple consists of *one* curve with a horizontal mark across the center (℈).

The dram consists of what may be considered *two* curves (ʒ).

The ounce consists of *three* curves (℥).

If the individual will note that the scruple, dram, and ounce are represented by *one* curve, *two* curves, and *three* curves respectively, he will have the solution of the difficulty.

(3). The Grain.

There should be in every school a pair of small balances with a set of apothecary weights to give the pupil the opportunity to learn the grain. There are small balances in the market—balances with beam 6 inches to 8 inches in length with brass pans 2 inches to 3 inches in diameter. Before the war, the cost including the weights was $1.25 to $1.50. Such balances may be obtained through almost any chemical supply house. They answer every purpose for the development of "grain" imagery—power to *estimate* in grains.

Additional grain weights (additional to those which come in a set of weights) may be secured by finding common pins that weigh even grains—one grain or two grains. Good quality pins of one kind are almost exactly equal to each other in weight. If the teacher succeeds in finding a paper of grain or 2-grain pins of such quality, she has in these pins as many additional grain weights for use on the balances as she chooses to purchase.

(4). Bottle Measurements.

In connection with the work upon the apothecary table the pupil should learn bottle measurement language.

"A pint is a pound
The world around."

is the common form in which the basis of this language is expressed.

The pupil should be taught that when the druggist speaks of an 8-ounce bottle he means a half-pint bottle, that his 4-ounce bottle holds ¼ of a pint, that his 12-ounce bottle holds ¾ of a pint. This pint-pound language is so common that the child should learn it. The teacher will bear in mind that the pound referred to is the avoirdupois pound. The basis of the language is the fact that a pint of distilled water at 39.2 Fahrenheit weighs approximately 16 ounces avoirdupois.

f. Circular Measure

The first use that the pupil has for this form of measurement is in the grammar school in mathematical geography. The table must be taken up in connection with that work.

The use of circular measure units in expressing the measure of *angles*—angles of 90°, of 45°, of 180°, etc.—should *follow* its use in expressing lengths of *arcs*. When the child is able to use the language intelligently as applied to *arcs*, he is ready to understand what is meant by *angles* of 90°, 45°, etc. The "arc" language must first be fully learned, afterwards the child will be able to understand "angle" language.

The introduction of the term degree should be with a circle drawn on the blackboard. An arc of 90° is marked off on this circle. "We call the distance from this point to this point 90 degrees." "What would this distance," extending the arc another 90°, "be called?" The inference is immediate that the distance from the beginning of the first arc to the end of the second would be 180°. The arc is now extended another 90°. The child is asked for the total length of the three arcs. This he will give as 270°. The total distance around the circle is now asked for. This brings out that the total distance is 360°. The same work is repeated with a much larger circle. This will bring out that on this circle, too, the corresponding distances are 90°, 180°, 270°, and 360°.

The process is repeated with a very small circle, then with a very large circle, then with a dozen circles of different diameters drawn on a common center. This gives a foundation for developing that—

90° means ¼ of the distance around a circle.
270° means ¾ of the distance around a circle.
The whole distance around a circle is 360°.
1° is 1/360 of the distance around a circle.

In connection with this other arcs must be brought in—45°, 22½°, 10°, 30°, etc.

The language is now applied to circles on the globe in connection with mathematical geography—to distances from one meridian to another, this distance being traced at one time along the equator, at another time along one of the tropics, at other times along others of the "parallels." It should be applied to lengths of meridians, to distances from equator to "parallels," to distances from one parallel to another, to distances from equator to poles.

The subdivisions of the degree (minutes and seconds) should be introduced later. The teacher must see that the signs ′ and ″ are used and that the pupil understands that "min." and "sec." are *time* and not circle expressions.

g. *Metric Units.*

The war has brought into almost common use in connection with the news from the fighting forces in Europe the terms meter and kilometer. In view of this the schools should supply the information that every pupil ought to have with regard to these units. He should become acquainted practically with the linear units millimeter, centimeter, meter, dekameter, and kilometer. He should know the comparative length of the meter (39.37 inches) and should have much practice in rapidly computing from the 39.37 inches the comparative lengths in inches of the decimeter, centimeter, and millimeter. He should compute and memorize the kilometer in yards and in the approximate fraction of a mile.

The metric table of weights should be learned and the comparative weights of the gram and kilogram in our grains and pounds should be memorized. These terms are so commonly used that the pupil should know the approximate values represented.

8. QUESTIONS FOR FREQUENT USE

Questions like the following should be used frequently covering all sorts of units in the activities with which the pupil comes in contact in his daily work. These questions are only suggestive. Local conditions and business activities determine what the questions should cover. A farming community would have units of measurement and things to be bought and sold that the child in a city knows little about. The reverse is true to some extent.

What is the unit in buying eggs? (Dozen).

What is the unit in buying apples? (Barrel, bushel).

What is the unit in buying butter?

What is the unit in measuring gold? (Grain, Troy ounce, Troy pound).

What is the unit in time measure?

What is the unit in U. S. money? (Dollar).

What is the unit in buying ribbon? (Linear yard).

What is the unit in buying cloth?

What is the unit in buying milk?

What is the unit in buying shoes?. (Pair).

What is the unit in buying hats? (The hat).

What is the unit in buying men's clothes? (Suit).

What is the unit in buying thread? (Spool).

What is the unit in buying buttons? (Dozen).

What is the unit in buying paper? (Quire, ream).

What is the unit in buying paint? (Gallon).

How do we buy wall paper? (By the roll).

How do we buy peanuts? (By the pint.).

How do we buy pencils? (By the pencil or dozen or gross).

How do we buy pens? (By the pen or dozen or gross).

How do we buy oranges? (By the dozen).

How do we buy candy? (By the pound).

How do we buy potatoes? (By the bushel or peck).

How do we buy coal? (By the ton).

There are weights and measures of common articles that the pupils should know. Among these are—

Pounds in a barrel of flour.

Pounds in a sack of flour. (1/8 bbl. sack).

Pounds in a bushel of potatoes.

Pounds in a peck of potatoes.

Pounds in a long ton, etc.

9. ESTIMATING.

In all objective denominate number subjects, there should be regular training in making estimates of quantitative values— estimates of small distances in inches or feet or yards; estimates of the capacities of small vessels such as fruit jars, other common cans, plant pots, etc.; estimates of weights, etc. This can will hold a pint, this plant-pot will hold about 2 quarts, this dish will hold about 1½ quarts, this is about two yards, this is about a rod, this stick is about 8 inches long. It is not necessary that the child should have to estimate measurements exactly. The concentration of attention that such work requires helps very greatly in the development of imagery. This is its aim.

The fact must not be lost sight of that work in estimating must not stop with the estimate. When the pupil makes a guess as to a quantity, that guess must be followed by an actual measurement or computation. The estimate must in every case be verified, otherwise much of the value that the work should have will not be realized.

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